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Hamilton formalism in non-commutative geometry ★

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Abstract

We study the Hamilton formalism for Connes–Lott models, i.e. for Yang–Mills theory in non-commutative geometry. The starting point is an associative $*$-algebra $\mathcal{A}$ which is of the form $\mathcal{A} = C(I, \mathcal{A}_s)$, where $\mathcal{A}_s$ is itself an associative $*$-algebra. With an appropriate choice of a K-cycle over $\mathcal{A}$ it is possible to identify the time-like part of the generalized differential algebra constructed out of $\mathcal{A}$. We define the non-commutative analogue of integration on space-like surfaces via the Dixmier trace restricted to the representation of the space-like part $\mathcal{A}_s$ of the algebra. Due to this restriction it is possible to define the Lagrange function resp. Hamilton function also for Minkowskian space–time. We identify the phase–space and give a definition of the Poisson bracket for Yang–Mills theory in non-commutative geometry. This general formalism is applied to a model on a two-point space and to a model on Minkowski space–time × two-point space.

Keywords: Connes–Lott models; Hamilton formalism; Yang–Mills theory; Non-commutative geometry
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1. Introduction

In the last few years it has turned out that Connes' non-commutative geometry provides a framework which allows for new qualitative insights in the spontaneous symmetry breaking mechanism of Yang–Mills theories. The cornerstone of this approach is a generalization of the algebra of differential forms and its corresponding differential. This has been used to construct models for the electroweak interaction [1,2] and Grand Unification [3,4]. Since the generalization of the differential algebra and its differential is not unique there are alternative models for the electroweak interaction, like the one developed by the Marseille and Mainz

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groups [5–7]. However, all models have in common that the Higgs field is interpreted as a part of the generalized connection form, although the precise form of the Higgs potential depends on the model chosen.

Another feature, which is common to all models so far, is that they are purely classical models, i.e. non-commutative geometry has been used to derive classical actions. In this approach some coupling constants, like the Higgs mass and the top mass in the Connes–Lott model, appear naturally restricted. However, such relations at the classical level cannot be translated to relations at the quantum level in an obvious way [8]. The reason for this is that it is not known so far how to quantize a theory in the framework of non-commutative geometry and for the usual quantization procedure it does not matter if some coupling constants of the classical action are fixed by hand or by some general principles of non-commutative geometry. Therefore it seems desirable to have a translation of the usual quantization procedure into the language of non-commutative geometry in order to get new insight in quantized Yang–Mills theory.

The generalization of geometry to non-commutative geometry is achieved by translating geometrical concepts into an algebraic language where conventional geometry corresponds to commutative algebras. The generalization is then obtained by extending those concepts to non-commutative algebras.

The quantization procedure which is closely related to algebra is the canonical quantization method. This approach to quantum theory is based on the Hamilton formalism. The purpose of this article is to develop a Hamilton formalism for (generalized) Yang–Mills theories in non-commutative geometry as they were introduced in [1,2].

This article is organized as follows. In Section 2 we give a motivation for the structure $\mathcal{A} = C(I, \mathcal{A}_s)$ of the associative $*$-algebra $\mathcal{A}$ which is the starting point for the derivation of Yang–Mills theory in non-commutative geometry. The universal differential enveloping algebra and the concept of finitely summable K-cycles are briefly reviewed in Section 3 where we construct a K-cycle which is appropriate for our purpose. In Section 4 the generalized differential algebra $\Omega_D\mathcal{A}$ of Connes [2] is discussed where we use the structure on $\mathcal{A}$ and the K-cycle, introduced in the previous sections, to show that there is a split of $\Omega_D\mathcal{A}$ into a "space-like" and a "time-like" part. The trace theorem of Connes [14] is used in Section 5 to define an inner product on $\Omega_D\mathcal{A}$. This definition differs from the usual definition in the sense that it corresponds to an integration on a "space-like" surface. As a consequence it is possible to define it also on space–time geometries with Minkowski metric. After a brief review of Yang–Mills theory in non-commutative geometry as it was introduced by Connes and Lott [1,2], the Lagrange function and the Hamiltonian for Yang–Mills theory are defined in Section 6. The formal construction ends with the definition of the Poisson bracket and time evolution in Section 7. In Section 8 the formalism is applied to two examples, namely to a discrete space and to Yang–Mills theory with symmetry breaking. The article ends with some conclusions in Section 9.

2. The algebra $\mathcal{A}$

Hamilton formalism is related to Cauchy surfaces in space–time and the separation of time which implies that the space–time manifold M has the topology

$$M = \mathbb{R} \times \Sigma, \tag{2.1}$$

where $\mathbb{R}$ corresponds to time and Σ to a (compact) space-like manifold. As a consequence the corresponding C^*-algebra of continuous functions (vanishing at infinity) $C_0(M)$ is of the form

$$C_0(M) = C_0(\mathbb{R}) \otimes C(\Sigma) = C_0(\mathbb{R}, C(\Sigma)), \tag{2.2}$$

where $C_0(\mathbb{R}) \otimes C(\Sigma)$ denotes the completion of the algebraic product of $C_0(\mathbb{R})$ and $C(\Sigma)$ and $C_0(\mathbb{R}, C(\Sigma))$, or more generally $C_0(\mathbb{R}, \mathcal{A})$, is the algebra of continuous functions over $\mathbb{R}$ with values in $C(\Sigma)$ resp. with values in some normed algebra $\mathcal{A}$.

The starting point of Connes generalization of differential forms is an associative $*$-algebra $\mathcal{A}$ (a subalgebra of a C^*-algebra). Eq. (2.2) motivates us to require that $\mathcal{A}$ has some additional structure which allows to introduce "time" to the formalism of generalized differential forms. Thus we postulate that

$$\mathcal{A} = C(I, \mathcal{A}_s), \tag{2.3}$$

where I is either $\mathbb{R}$ or S^1 and $\mathcal{A}_s$ is a normed associative $*$-algebra with unit, possessing a finitely summable K-cycle. If $\mathcal{A}_s$ is a C^*-algebra we have

$$\mathcal{A} = C(I, \mathcal{A}_s) = C(I) \otimes \mathcal{A}_s, \tag{2.4}$$

where $C(I) \otimes \mathcal{A}_s$ again denotes the completion of the algebraic product of $C(I)$ and $\mathcal{A}_s$. Since $\mathcal{A}_s$ has a unit we can identify $C(I)$, the algebra of continuous functions on I, as a subalgebra of $\mathcal{A}$ by

$$i_t : C(I) \longrightarrow \mathcal{A}, \qquad i_t(f) = f \otimes 1_s, \quad f \in C(I), \tag{2.5}$$

where 1_s denotes the unit element in $\mathcal{A}_s$.

We shall assume that $\mathcal{A}$ has a unit element. If I is compact (i.e. $I = S^1$) then $C(I)$ and therefore also $\mathcal{A}$ has a unit element. If $I = \mathbb{R}$ then $C_0(I)$ does not have a unit. However we can always formally add a unit element to $C_0(I)$ which induces a unit element in $\mathcal{A}$. Furthermore we can use the unit element 1_t of $C(I)$ to identify $\mathcal{A}_s$ as a subalgebra of $\mathcal{A}$:

$$i_s : \mathcal{A}_s \longrightarrow \mathcal{A}, \qquad i_s(a) = 1_t \otimes a, \quad f \in \mathcal{A}_s. \tag{2.6}$$

3. The universal differential envelope and K-cycles over $\mathcal{A}$

In this and the subsequent section we follow Connes construction of generalized differential forms [2]. However, we will focus on the structure of $\mathcal{A} = C(I, \mathcal{A}_s)$ which will lead to a "time-split" in the generalized differential algebra. For details of the general construction we refer to [2,9,10].

The first step is to construct a bigger algebra $\Omega\mathcal{A}$ by associating to each element $A \in \mathcal{A}$ a symbol δA. $\Omega\mathcal{A}$ is the free algebra generated by the symbols A, δA, $A \in \mathcal{A}$ modulo the relation

$$\delta(AB) = \delta A\, B + A\, \delta B. \tag{3.1}$$

With the definition

$$\delta(A_0 \delta A_1 \cdots \delta A_k) := \delta A_0\, \delta A_1 \cdots \delta A_k, \qquad \delta(\delta A_1 \cdots \delta A_k) := 0 \tag{3.2}$$

$\Omega\mathcal{A}$ becomes a $\mathbb{N}$-graded differential algebra with the odd differential δ, $\delta^2 = 0$. $\Omega\mathcal{A}$ is called the universal differential envelope of A. By defining

$$\delta(A)^* = -\delta(A^*), \tag{3.3}$$

the $*$-operation is extended uniquely to $\Omega\mathcal{A}$.

The next element in the construction is the K-cycle $(\mathcal{H}, D)$ over $\mathcal{A}$. It consists of a Hilbert space $\mathcal{H}$ with a faithful $*$-representation π,

$$\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H}), \tag{3.4}$$

where $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded operators on $\mathcal{H}$. The second part of the K-cycle is an unbounded self-adjoint operator D on $\mathcal{H}$. Since the K-cycle should also reflect the structure given by Eq. (2.3) let us first discuss the representation π a little bit further before we come to structure of D. However, the main strategy will be to construct the K-cycle $(\mathcal{H}, D)$ over $\mathcal{A}$ out of K-cycles $(\mathcal{H}_s, D_s)$ over $\mathcal{A}_s$.

Suppose $\mathcal{H}_s$ is a (separable) Hilbert space with an inner product $(\cdot,\cdot)_s$ and a faithful $*$-representation $\tilde{\pi}_s$,

$$\tilde{\pi}_s : \mathcal{A}_s \longrightarrow \mathcal{B}(\mathcal{H}_s). \tag{3.5}$$

$\mathcal{H}_s$ can be extended to a bigger Hilbert space

$$\mathcal{H} = L_2(I, \mathcal{H}_s) \tag{3.6}$$

with the inner product

$$(\Psi, \Phi) = \int_I \mathrm{d}t (\Psi(t), \Phi(t))_s. \tag{3.7}$$

The representation $\tilde{\pi}_s$ on $\mathcal{H}_s$ induces a representation π_s on $\mathcal{H}$ of $\mathcal{A}_s$,

$$\pi_s : \mathcal{A}_s \longrightarrow C(I, \mathcal{B}(\mathcal{H}_s)) \subset \mathcal{B}(\mathcal{H}), \tag{3.8}$$

by identifying $\mathcal{B}(\mathcal{H}_s)$ with the subalgebra of operators in $C(I, \mathcal{B}(\mathcal{H}_s))$ which are constant in $t \in I$. There is also a representation π_t of $C(I)$,

$$\pi_t : C(I) \longrightarrow C(I, \mathcal{B}(\mathcal{H}_s)), \qquad \pi_t(f) = f\,\mathrm{id}_s, \quad f \in C(I), \tag{3.9}$$

where id_s denotes the unit element in $\mathcal{B}(\mathcal{H}_s)$. Because of Eq. (2.3) these two representations induce a faithful $*$-representation π of $\mathcal{A}$,

$$\pi : \mathcal{A} \longrightarrow C(I, \mathcal{B}(\mathcal{H}_s)) \tag{3.10}$$

with

$$\pi(f \otimes a) = \pi_t(f)\pi_s(a) = \pi_s(a)\pi_t(f), \quad f \otimes a \in \mathcal{A}. \tag{3.11}$$

Strictly speaking, π_s and π_t define a representation of a dense subalgebra of $\mathcal{A}$, which can be extended to a representation of $\mathcal{A}$.

Let us now turn to the second element of the K-cycle, the operator D on $\mathcal{H}$. The general conditions to be fulfilled by this operator are [2]:

(i) D is self-adjoint;
(ii) $[D, \pi(A)]$ is a bounded operator;
(iii) D is unbounded with a compact inverse (modulo finite rank operators) such that $|D|^{-1}$ is d^+ summable for some $d \in \mathbb{N}$;

If $\mathcal{A}$ is a C^*-algebra condition (ii) holds only on a dense subalgebra of $\mathcal{A}$ in general. Therefore we denote in the following by $\mathcal{A}$ a dense subalgebra of a C^*-algebra such that (iii) holds for any element of $\mathcal{A}$, i.e. $\mathcal{A} = C^\infty(I, \mathcal{A}_s)$, where $\mathcal{A}_s$ is also a suitable dense subalgebra of a C^*-algebra. However, since D is closely related to the metric structure of the underlying manifold, which is also the case for non-commutative geometries [2], [1] we have to impose further conditions on D. They should reflect the topology which is encoded in the structure (2.3) of $\mathcal{A}$. This motivates the additional requirement that D is the sum of two operators

$$D = D_t + D_s \tag{3.12}$$

with

(iv) $[D_t, \pi_s(a)] = 0$ and $[D_s, \pi_t(f)] = 0, \ \forall f \in C(I), \ \forall a \in \mathcal{A}_s$;
(v) $[D_t, \pi_t(f)][D_s, \pi_s(a)] + [D_s, \pi_s(a)][D_t, \pi_t(f)] = 0, \ \forall f \in C(I), \ \forall a \in \mathcal{A}_s$;
(vi) $D_s \in C^\infty(I, \mathcal{O}_s)$ [2] (where $\mathcal{O}_s$ denotes the algebra of operators on $\mathcal{H}_s$) thus D_s is as a smooth 1-parameter family of operators on $\mathcal{H}_s$. $D_s(t)$ fulfills conditions (i)–(iii) with $\mathcal{A}$ replaced by $\mathcal{A}_s$, $\mathcal{H}$ replaced by $\mathcal{H}_s$. In other words $(\mathcal{H}_s, D_s(t))$ is a smooth 1-parameter family of K-cycles over $\mathcal{A}_s$.

We now show how a K-cycle $(\mathcal{H}, D)$ over $\mathcal{A}$ can be constructed out of a 1-parameter family of K-cycles $(\tilde{\mathcal{H}}_s, \tilde{D}_s(t))$, $t \in I$, over $\mathcal{A}_s$. Having the 3 + 1-dimensional case in mind, we do not assume that there is a grading on $\mathcal{H}_s$, i.e. an automorphism γ with $\gamma^2 = 1$

[1] In fact, if D is a Dirac operator it is possible to construct a gravity-action by taking the Wodzicki residue of an appropriate inverse power of D [11–13].
[2] Note, this implies the second part of (iv).

and $[\gamma, \tilde{D}_s]_+ = 0$[3] and $[\gamma, \pi_s] = 0$. However, a substitute for this automorphism can always be constructed by extending $\tilde{\mathcal{H}}_s$, which also allows to drop the condition that $\tilde{D}_s$ is self-adjoint. We extend the Hilbert space by $\mathbb{C}^2$:

$$\mathcal{H}_s = \tilde{\mathcal{H}}_s \otimes \mathbb{C}^2, \qquad \pi_s = \tilde{\pi}_s \otimes 1_{\mathbb{C}^2} \tag{3.13}$$

and

$$D_s = \begin{pmatrix} 0 & \tilde{D}_s \\ \tilde{D}_s^{\dagger} & 0 \end{pmatrix}. \tag{3.14}$$

The Hilbert space $\mathcal{H}_s$ and the representation π_s can be extended to a Hilbert space $\mathcal{H}$ and a representation π of $\mathcal{A}$ in the above mentioned manner. What is still missing is the operator D. More precisely the part D_t of D has to be specified. A natural choice is $D_t \sim \partial_t$. However, condition (v) has to be taken into account. Therefore we introduce an element $\gamma^0 \in \mathcal{B}(\mathcal{H})$ (a substitute for the grading) with the following properties:

$$\gamma^0 = \begin{pmatrix} \tilde{\gamma} & 0 \\ 0 & -\tilde{\gamma} \end{pmatrix}; \tag{3.15}$$

$$\begin{aligned} &(\gamma^0)^2 = \pi(N), \quad N \in \mathcal{A}; \\ &[\gamma^0, \pi(A)] = 0, \quad \forall A \in \mathcal{A}; \qquad (\gamma^0)^{-1} \text{ exists} \end{aligned} \tag{3.16}$$

and

$$\gamma^{0\dagger} = -\gamma^0, \tag{3.17}$$

where the same block structure as in Eq. (3.13) is used. Such an element always exists since $\mathcal{A}$ has a unit element. We now define D_t by

$$D_t = \gamma^0 \partial_t \ . \tag{3.18}$$

The anti self-adjointness of γ^0 (Eq. (3.17)) ensures the self-adjointness of $D = D_t + D_s$. It is straightforward to check that for this choice of $D = D_t + D_s$, with D_t resp. D_s defined as in Eq. (3.18) resp. (3.14), and $\mathcal{H}$ $(\mathcal{H}, D)$ is a K-cycle over $\mathcal{A} = C^\infty(I, \mathcal{A}_s)$ which fulfills conditions (ii) and (iv)–(vi). Condition (iii) is not needed for the definition of the generalized differential algebra. It is crucial for the definition of the operator theoretic substitute for integration. However, since we are only interested in a substitute for integration on "space-like" surfaces we can replace this condition on D by an analogous condition on D_s (vi). However, this condition on D and the self-adjointness of D is related to the Euclidean signature of the metric of the underlying manifold. As we shall see in Section 4, with the choice $\gamma^0 = \gamma^{0\dagger}$ one obtains an operator D corresponding to an underlying manifold with Minkowski metric.

[3] $[\cdot, \cdot]_+$ denotes the anti-commutator.

4. The generalized differential algebra

Having introduced the generalized differential algebra $\Omega\mathcal{A}$ of $\mathcal{A}$ and a K-cycle $(\mathcal{H}, D)$ over $\mathcal{A}$ we can now put these elements together in order to define a generalized differential algebra as it was introduced by Connes [2]. We begin by extending the $*$-representation π to a $*$-representation of the algebra $\Omega\mathcal{A}$:

$$\begin{aligned}&\pi_D : \Omega_D \longrightarrow \mathcal{B}(\mathcal{H}),\\ &\pi_D(A_0\delta A_1 \cdots \delta A_k) = \pi(A_0)[D,\pi(A_1)]\cdots[D,\pi(A_k)].\end{aligned} \tag{4.1}$$

However, there is another possibility to extend π to a representation of $\Omega\mathcal{A}$ which is useful for our purpose:

$$\begin{aligned}&\pi_{D_s} : \Omega_D \longrightarrow \mathcal{B}(\mathcal{H})\\ &\pi_{D_s}(A_0\delta A_1 \cdots \delta A_k) = \pi(A_0)[D_s,\pi(A_1)]\cdots[D_s,\pi(A_k)].\end{aligned} \tag{4.2}$$

Obviously the kernel of π_{D_s} is much bigger than the kernel of π_D. For instance $\delta C^\infty(I) \subset \Omega^1\mathcal{A}$ is contained in the kernel of π_{D_s} because only the "space-like" part D_s of D is used in the definition (Eq. (4.2)) of π_{D_s}.

On the images of π_D resp. π_{D_s} the differential δ on $\Omega\mathcal{A}$ does not induce well-defined differentials. Therefore one has to divide out two-sided graded differential ideals. For π_D such an ideal is given by

$$\mathcal{J}^k = \ker\pi_D \cap \Omega^k\mathcal{A} + \delta(\ker\pi_D \cap \Omega^{k-1}\mathcal{A}), \qquad \mathcal{J} = \bigoplus_{k=1}^{\infty}\mathcal{J}^k. \tag{4.3}$$

On the quotient algebra $\Omega_D\mathcal{A}$, which is defined as

$$\Omega_D^k\mathcal{A} = \frac{\Omega^k\mathcal{A}}{\mathcal{J}^k}, \qquad \Omega_D\mathcal{A} = \bigoplus_{k=1}^{\infty}\Omega_D^k\mathcal{A}, \tag{4.4}$$

there is a differential d with $d^2 = 0$ which is uniquely defined by the differential δ on $\Omega\mathcal{A}$ as

$$d(\sigma_D\pi_D(\omega)) = \sigma_D\pi_D(\delta\omega), \quad \omega \in \Omega_D, \tag{4.5}$$

where σ_D denotes the map

$$\sigma_D : \pi_D(\Omega^k\mathcal{A}) \longrightarrow \Omega_D^k\mathcal{A}. \tag{4.6}$$

Thus $\Omega_D\mathcal{A}$ is a generalized graded differential algebra [2].

Of course, it is now possible to define the differential ideal associated to π_{D_s} in a completely analogous way as for π_D. This would also lead to a differential algebra with a differential which is uniquely defined by the differential on $\Omega\mathcal{A}$. However, such a differential algebra would not have an interpretation as the "space-like" part of $\Omega_D\mathcal{A}$ in general. Therefore one has to divide out a bigger differential ideal. One is led to the correct ideal by the two lemmas following the next little preparing lemma.

Lemma 1. *For $I = S^1$ there is an*

$$\eta = \sum_i f^{(i)} \delta g^{(i)} \in \Omega^1 \mathcal{A}, \quad f^{(i)}, g^{(i)} \in C(I) \tag{4.7}$$

such that

$$\pi_D(\eta) = \gamma^0. \tag{4.8}$$

If $I = \mathbb{R}$ there is a sequence

$$\eta_n = \sum_i f_n^{(i)} \delta g_n^{(i)} \in \Omega^1 \mathcal{A}, \quad f_n^{(i)}, g_n^{(i)} \in C(I) \tag{4.9}$$

such that

$$\lim_{n\to\infty} \pi_D(\eta_n) = \gamma^0 \tag{4.10}$$

in the strong operator topology of $\mathcal{B}(\mathcal{H})$.

Proof. If $I = S^1$ let U_i be a finite open cover of S^1, f_i the corresponding partition of unity and g_i some smooth functions on S^1 with $\partial_t g_i = 1, \forall t \in U_i$ then η defined as in Eq. (4.7) fulfills Eq. (4.8).

If $I = \mathbb{R}$ there are no bounded functions in $C_0^\infty(\mathbb{R})$ such that η can be defined as in Eq. (4.7). However let $\{a_n\}_{n\in\mathbb{N}}, a_{n+1} > a_n > 0$ be a sequence in $\mathbb{R}$ with $a_n \to \infty$ as $n \to \infty$. Define $U_n =]-a_n, a_n[$ and choose $f_n, g_n \in C_0^\infty(\mathbb{R})$ such that $f_n(t) = 1,\ t \in U_n$, $|f_n(t)| \le 1,\ t \notin U_n$ and $\partial_t g_n(t) = 1,\ t \in U_n,\ |\partial_t g_n(t)| \le 1,\ t \notin U_n$. For $\eta_n = f_n \delta g_n$ it is

$$\begin{aligned} (\Psi, (\pi_D(\eta_n) - \gamma^0)\Psi) &= \int_{\mathbb{R}} \mathrm{d}t (f_n \partial_t g_n - 1)(\Psi, \gamma^0 \Psi)_s \\ &\le \int_{\mathbb{R}} \mathrm{d}t (\Psi, \Psi)_s - \int_{-a_n}^{a_n} \mathrm{d}t (\Psi, \Psi)_s, \quad \Psi \in \mathcal{H}. \end{aligned} \tag{4.11}$$

Thus $\pi_D(\eta_n)$ converges in the strong operator topology to γ^0. □

If $I = \mathbb{R}$ we add a formal limit point η of the sequence η_n to $\Omega^1 \mathcal{A}$ with

$$\pi_D(\eta) = \gamma^0. \tag{4.12}$$

This element is formal since we have not specified a topology on Ω_D which would allow to consider convergence of η_n in Ω_D. However, except for the definition of the map T, the element η will appear only as an argument of π_D or π_{D_s} and therefore the limit is well defined in $\mathcal{B}(\mathcal{H})$. Furthermore we note that

$$\pi_{D_s}(\eta) = 0. \tag{4.13}$$

Lemma 2. *For $\omega \in \Omega^k \mathcal{A}$ we have*

$$\pi_D(\omega) = \pi_{D_s}(\omega) + \pi_D(\alpha), \quad \alpha \in \ker \pi_{D_s} \cap \Omega^k \mathcal{A}. \tag{4.14}$$

Proof. We prove this lemma by defining an algebra homomorphism T on $\Omega\mathcal{A}$ which is an projection, i.e. $T^2 = T$. $\Omega\mathcal{A}$ is generated by the zeroth and first degree and therefore it is sufficient to define T on those spaces. Since $\mathcal{A} = C^\infty(I, \mathcal{A}_s)$ it is $\partial_t A \in \mathcal{A}\ \forall A \in \mathcal{A}$. We use this and the element η to define

$$\begin{aligned} &T(A) = A, \quad A \in \mathcal{A}, \\ &T(\delta A) = \delta A - \partial_t A\eta; \qquad T(\eta) = 0, \quad A \in \mathcal{A}. \end{aligned} \tag{4.15}$$

For an arbitrary degree $k > 1$ one obtains

$$T : \Omega^k\mathcal{A} \longrightarrow \Omega^k\mathcal{A}, \qquad T(A_0\delta A_1 \cdots \delta A_k) = A_0 T(\delta A_1) \cdots T(\delta A_k). \tag{4.16}$$

Since

$$\pi_D(\delta A) = [D_s, \pi(A)] + [D_t, \pi(A)] = \pi_{D_s}(\delta A) + \pi_D(\partial_t A\eta), \tag{4.17}$$

this map has the useful property that for any $\omega \in \Omega\mathcal{A}$

$$\pi_{D_s}(\omega) = \pi_D(T(\omega)), \qquad \pi_{D_s}((1 - T)\omega) = 0. \tag{4.18}$$

It is $\pi_D(\Omega^0\mathcal{A}) = \pi_{D_s}(\Omega^0\mathcal{A}) = \pi(\mathcal{A})$ and for any k it is

$$\pi_D(\omega) = \pi_D(T(\omega)) + \pi_D((1 - T)\omega) = \pi_{D_s}(\omega) + \pi_D(\alpha), \quad \forall\omega \in \Omega^k\mathcal{A} \tag{4.19}$$

with $\alpha = (1 - T)(\omega) \in \ker\pi_{D_s}$ and the lemma is proved. □

Lemma 3. *It is*

$$\pi_D(\mathcal{A}) \subset \pi_D(\mathcal{J}_D^2) \subset \pi_D(\Omega^2\mathcal{A}) \tag{4.20}$$

and thus there is a filtration on $\pi_D(\Omega\mathcal{A})$:

$$\begin{aligned} &\pi_D(\Omega^0\mathcal{A}) \subset \pi_D(\Omega^2\mathcal{A}) \subset \pi_D(\Omega^4\mathcal{A}) \subset \cdots \\ &\pi_D(\Omega^1\mathcal{A}) \subset \pi_D(\Omega^3\mathcal{A}) \subset \pi_D(\Omega^5\mathcal{A}) \subset \cdots \end{aligned} \tag{4.21}$$

Proof. Let us consider $\alpha = (f \otimes 1)\delta(g \otimes 1) + (g \otimes 1)\delta(f \otimes 1) - \delta(fg \otimes 1) \in \Omega^1\mathcal{A}$. It is

$$\pi_D(\alpha) = 0 \tag{4.22}$$

but

$$\pi_D(\delta\alpha) = 2[D_t, \pi_t(f)][D_t, \pi_t(g)] = 2(\gamma^0)^2\partial_t f\partial_t g = \pi(N)\pi(\partial_t f\partial_t g) \in \pi(\mathcal{A}), \tag{4.23}$$

i.e. $\delta\alpha \in \mathcal{J}_D^2$ and with such elements all of $\pi(\mathcal{A})$ can be generated. Thus the lemma is proved. □

Suppose $\omega \in \ker\pi_D \cap \Omega^k\mathcal{A}$. Lemma 2 allows us to write

$$0 = \pi_D(\omega) = \pi_{D_s}(\omega) + \pi_D(\alpha). \tag{4.24}$$

Because of Lemma 3 we cannot infer that $\pi_D(\omega) = 0$ implies $\pi_{D_s}(\omega) = 0$. Thus if we divide $\Omega\mathcal{A}$ by the differential ideal associated to π_{D_s} in an analogous way as for π_D in Eq. (4.3) it may happen that the resulting differential algebra contains elements which are not elements of $\Omega_D\mathcal{A}$ and therefore it is not a subalgebra of $\Omega_D\mathcal{A}$. The correct differential ideal, which leads to a graded subalgebra of $\Omega_D\mathcal{A}$ is constructed with the help of the following ideal in $\Omega\mathcal{A}$:

$$\mathcal{K}^{2k} = \left\{\omega \in \Omega^{2k}\mathcal{A} \mid \exists\alpha \in \bigoplus_{j=0}^{k-1} \Omega^{2j}\mathcal{A}\,,\ \pi_{D_s}(\omega+\alpha) = 0\right\},$$
$$\mathcal{K}^{2k+1} = \left\{\omega \in \Omega^{2k+1}\mathcal{A} \mid \exists\alpha \in \bigoplus_{j=0}^{k-1} \Omega^{2j+1}\mathcal{A}\,,\ \pi_{D_s}(\omega+\alpha) = 0\right\}, \tag{4.25}$$
$$\mathcal{K} = \bigoplus_{k=1}^{\infty} \mathcal{K}^k.$$

Let us also define the ideal $\mathcal{K}_0$

$$\mathcal{K}_0^k = \ker \pi_{D_s} \cap \Omega^k\mathcal{A}, \qquad \mathcal{K}_0 = \bigoplus_{k=1}^{\infty} \mathcal{K}_0^k. \tag{4.26}$$

A two sided differential ideal $\mathcal{N}$ is obtained as in Eq. (4.3) by including the image of δ on $\mathcal{K}$:

$$\mathcal{N}^k = \mathcal{K}^k + \delta\mathcal{K}^{k-1}, \qquad \mathcal{N} = \bigoplus_{k=1}^{\infty} \mathcal{N}^k. \tag{4.27}$$

The corresponding graded differential algebra $\Omega_{\mathcal{N}}\mathcal{A}$ is then defined as

$$\Omega_{\mathcal{N}}{}^k\mathcal{A} = \frac{\Omega^k\mathcal{A}}{\mathcal{N}^k}, \qquad \Omega_{\mathcal{N}} = \bigoplus_{k=0}^{\infty} \Omega_{\mathcal{N}}{}^k. \tag{4.28}$$

Let us denote by $\sigma_{\mathcal{N}}$, the map on the quotient space,

$$\sigma_{\mathcal{N}} : \pi_{D_s}(\Omega^k\mathcal{A}) \longrightarrow \Omega_D^k\mathcal{A}. \tag{4.29}$$

The relation of $\Omega_D\mathcal{A}$ and $\Omega_{\mathcal{N}}\mathcal{A}$ is determined by the relation of $\mathcal{N}$ and $\mathcal{J}$ and therefore it is useful to prove the following lemma.

Lemma 4. *It is*

$$\mathcal{K}^k = (\ker \pi_D \cap \Omega^k\mathcal{A}) \cup \mathcal{K}_0^k \tag{4.30}$$

and

$$\mathcal{N}^k = \mathcal{J}^k \cup \mathcal{K}_0^k. \tag{4.31}$$

Proof. It is clear that $(\ker \pi_D \cap \Omega^k\mathcal{A}) \cup \mathcal{K}_0^k \subset \mathcal{K}^k$. Thus we have to consider elements $\omega \in \mathcal{K}^{2k}$ with

$$0 \neq \pi_{D_s}(\omega) = \sum_{j=0}^{k-1} \pi_{D_s}(\omega_{2j}), \quad \omega_{2j} \in \Omega^{2j}\mathcal{A}. \tag{4.32}$$

Because of Lemma 2 there are $\alpha_{2j} \in \mathcal{K}_0^{2j}$ and $\alpha \in \mathcal{K}_0^{2k}$ with

$$\pi_D(\omega_{2j} - \alpha_{2j}) = \pi_{D_s}(\omega_{2j}), \qquad \pi_D(\omega - \alpha) = \pi_{D_s}(\omega). \tag{4.33}$$

We define $\omega' \in \Omega^{2k}\mathcal{A}$ as

$$\omega' = \omega - \alpha - \sum_{j=0}^{k-1} (N^{-1}\eta)^{2(k-j)}(\omega_{2j} - \alpha_{2j}). \tag{4.34}$$

Since

$$\pi_{D_s}(\omega - \omega') = 0 \tag{4.35}$$

and

$$\pi_D(\omega') = 0 \tag{4.36}$$

we infer that $\omega \in (\ker \pi_D \cap \Omega^{2k}\mathcal{A}) \cup \mathcal{K}_0^{2k}$. The same is true for $\omega \in \mathcal{K}^{2k+1}$ and therefore Eq. (4.30) is proved.

For the second part of the proof we compute $[\delta, T]$:

$$\begin{aligned} \delta T(A_0 \delta A_1 \cdots \delta A_k) &= \delta(A_0 T(\delta A_1) \cdots T(\delta A_k)) \\ &= \delta A_0 T(\delta A_1) \cdots T(\delta A_k) \\ &\quad + \sum_{j=1}^{k} A_0 T(\delta A_1) \cdots (\delta A_j + (-1)^j \delta(\partial_t A_j \eta)) \cdots T(\delta A_k), \\ T\delta(A_0 \delta A_1 \cdots \delta A_k) &= T(\delta A_0) T(\delta A_1) \cdots T(\delta A_k). \end{aligned} \tag{4.37}$$

Thus

$$\begin{aligned} [\delta, T](A_0 \delta A_1 \cdots \delta A_k) &= \partial_t A_0 \eta T(\delta A_1) \cdots T(\delta A_k) \\ &\quad + \sum_{j=1}^{k} A_0 T(\delta A_1) \cdots (\delta A_j + (-1)^j \delta(\partial_t A_j \eta)) \cdots T(\delta A_k). \end{aligned} \tag{4.38}$$

Therefore, with $\pi_{D_s}(\delta\eta) = 0$, we conclude that for any $\omega \in \mathcal{K}_0$

$$\pi_{D_s}([\delta, T](\omega)) = 0. \tag{4.39}$$

Furthermore it is for $\omega \in \mathcal{K}_0$

$$0 = \pi_{D_s}(\omega) = \pi_D(T\omega) \tag{4.40}$$

and therefore

$$\pi_D(\delta T\omega) \in \pi_D(\mathcal{J}). \tag{4.41}$$

On the other hand it is

$$\pi_{D_s}(\delta\omega) = \pi_D(T\delta\Omega). \tag{4.42}$$

Together with Eq. (4.39) this proves Eq. (4.31). □

We now state the main result of this section which shows that $\Omega_{\mathcal{N}}\mathcal{A}$ is the "space-like" part of $\Omega_D\mathcal{A}$ in the sense that there is a "time" differential d_t and a "time-like" differential one-form dt in $\Omega_D\mathcal{A}$. We then denote by "space-like" forms such elements in $\Omega_D\mathcal{A}$ which do not contain dt.

Theorem 1. *There is an element $dt \in \Omega_D^1\mathcal{A}$ such that for any k*

$$dt\omega - (-1)^k\omega dt = 0 \quad \forall\omega \in \Omega_D^k \tag{4.43}$$

and

$$\Omega_D^k\mathcal{A} = \Omega_{\mathcal{N}}^k\mathcal{A} \oplus \Omega_{\mathcal{N}}^{k-1}\mathcal{A}dt. \tag{4.44}$$

The differential d on $\Omega_D\mathcal{A}$ is given as a sum of the two differentials d_s and d_t:

$$d = d_s + d_t, \tag{4.45}$$

$$d_t(\sigma_D\pi_D(\omega)) = (-1)^k\sigma_D\pi_D(T(\partial_t\omega))dt, \quad \omega \in \Omega^k\mathcal{A} \tag{4.46}$$

with

$$\begin{aligned}\partial_t(A_0\delta A_1\cdots\delta A_k) &= \partial_t A_0\delta A_1\cdots\delta A_k \\ &\quad + \sum_{j=1}^{k}(-1)^{k-j}A_0\delta A_1\cdots\delta(\partial_t A_j)\cdots\delta A_k.\end{aligned} \tag{4.47}$$

Proof. From Lemma 2 we know that $\pi_{D_s}(\Omega\mathcal{A})$ is a subalgebra of $\pi_D(\Omega\mathcal{A}) = \pi_{D_s}(\Omega\mathcal{A}) \cup \pi_D(\mathcal{K}_0)$ and hence

$$\bigoplus_{k=0}^{\infty}\frac{\pi_{D_s}(\Omega^k\mathcal{A})}{\pi_D(\mathcal{J}^k)} \subset \Omega_D\mathcal{A} \tag{4.48}$$

is a subalgebra of $\Omega_D\mathcal{A}$. Because of Eq. (4.31) we can conclude that

$$\bigoplus_{k=0}^{\infty}\frac{\pi_{D_s}(\Omega^k\mathcal{A})}{\pi_D(\mathcal{J}^k)} = \bigoplus_{k=0}^{\infty}\frac{\pi_{D_s}(\Omega^k\mathcal{A})}{\pi_D(\mathcal{N}^k)} = \Omega_{\mathcal{N}}\mathcal{A}. \tag{4.49}$$

From Eq. (4.30) we infer that

$$\frac{\pi_{D_s}(\Omega^k\mathcal{A})}{\pi_D(\mathcal{J}^k)} \bigcap \frac{\pi_D(\mathcal{K})}{\pi_D(\mathcal{J}^k)} = \{0\}. \tag{4.50}$$

Thus we can decompose $\Omega_D\mathcal{A}$ as follows:

$$\Omega_D^k\mathcal{A} = \Omega_{\mathcal{N}}^k\mathcal{A} \oplus \frac{\pi_D(\mathcal{K}_0^k)}{\pi_D(\mathcal{J}^k)}. \tag{4.51}$$

We proceed by identifying dt as

$$dt = \sigma_D \pi_D(\eta). \tag{4.52}$$

Because of Lemma 3 we know that $\eta^2 \in \mathcal{J}^2$ and hence $dt^2 = 0$. For any $A_0\delta A_1 \cdots A_k \in \mathcal{K}_0^k$ it is

$$\begin{aligned}\pi_D(A_0\delta A_1 \cdots \delta A_k) &= \pi_D(A_0)([D_s, \pi_D(A_1)] + \pi_D(\partial_t A_1)\pi_D(\eta)) \cdots \\ &\quad \cdots ([D_s, \pi_D(A_k)] + \pi_D(\partial_t A_k)\pi_D(\eta)) \qquad (4.53)\\ &= \sum_{j=1}^{k} (-1)^{k-j} \pi_{D_s}(A_0\delta A_1 \cdots \partial_t A_j \cdots \delta A_k)\pi_D(\eta) + \pi_D(\alpha),\end{aligned}$$

where $\pi_D(\alpha) \in \mathcal{J}^k$ denotes the sum of terms with a factor $\pi_D(\eta)^k$, $k > 1$. We also used condition (v) of D to anti-commute $\pi_D(\eta)$ to the right. From Eq. (4.54) we infer that

$$\frac{\pi_D(\mathcal{K}^k)}{\pi_D(\mathcal{J}^k)} = \Omega_{\mathcal{N}}^{k-1}\mathcal{A}dt. \tag{4.54}$$

Eq. (4.43) is also a consequence of condition (v) of D.

Since $\Omega_D\mathcal{A}$ is generated by $\Omega_D^1\mathcal{A}$ and $dt^2 = 0$ it is sufficient to show Eq. (4.46) for all $\mu \in \Omega_D^1\mathcal{A}$. For any $w \in \Omega_{\mathcal{N}}^1\mathcal{A}$, let $A_0\delta A_1 \in \Omega^1\mathcal{A}$ be a representative, i.e.

$$\sigma_{\mathcal{N}}\pi_{D_s}(A_0\delta A_1) = w. \tag{4.55}$$

Let us first compute the action of d_s on $\sigma_D\pi_D(T(A_0\delta A_1))$, which is the image of w in $\Omega_D\mathcal{A}$,

$$\begin{aligned}d_s\sigma_D\pi_D(T(A_0\delta A_1)) &= \sigma_D\pi_D(T(\delta A_0\delta A_1))\\ &= \sigma_D([D_s, \pi(A_0)][D_s, \pi(A_1)]).\end{aligned} \tag{4.56}$$

We use this to compute the action of d on $\sigma_D(A_0[D_s, \pi(A_1)])$

$$\begin{aligned}&d\sigma_D\pi_D(A_0\delta A_1)\\ &= \sigma_D\pi_D(\delta(A_0T(\delta A_1)))\\ &= \sigma_D(([D_s, \pi(A_0)] + [D_t, \pi(A_0)])\pi_D(T(\delta A_1))) + \sigma_D\pi_D(A_0\delta T(\delta A_1))\\ &= d_s\sigma_D\pi_D(A_0\delta A_1) + \sigma_D\pi_D(\partial_t A_0 T(\delta A_1)) + \sigma_D\pi_D(T(A_0\delta(\partial_t A_1)))dt.\end{aligned} \tag{4.57}$$

This shows that

$$d_t(\sigma_D\pi_D(A_0\delta A_1)) = (d - d_s)\sigma_D\pi_D(T(A_0\delta A_1)) = \sigma_D\pi_D(\partial_t(A_0\delta A_1)) \tag{4.58}$$

and the theorem is proved. □

5. The inner product on $\Omega_D\mathcal{A}$ and the Lorentz metric

So far we have constructed a generalized differential algebra where we were able to identify the "space-like" and the "time-like" part because of the structure $\mathcal{A} = C^\infty(I, \mathcal{A}_s)$

of the algebra and the special form of the K-cycle $(\mathcal{H}, D)$. Following the lines presented by Connes and Lott [1,2] it is now straightforward to reconstruct connection and curvature in this generalized non-commutative framework. However, there is still one important ingredient missing which is neccessary to define an action or a Lagrange function resp. a Hamilton function, the objects we are interested in. In conventional geometry one obtains an action or Lagrange function by integration over appropriate differential forms. In [14] Connes showed that the correct substitute for integration in non-commutative geometry is the Dixmier trace. It is this trace which is used in the definition of actions in [1,3,4]. However, we want to derive a Hamilton function and therefore we do not have to integrate over the non-commutative "space-time" but we have to integrate over a "space-like" surface. As before we will use the additional structure of $\mathcal{A}$ and $(\mathcal{H}, D)$ to define the correct operator theoretic substitute for integration on "space-like" surfaces which will be the Dixmier trace on $\mathcal{H}_s$.

Let us first briefly recall the definition of the Dixmier trace and some general results about the inner product on $\Omega_D\mathcal{A}$ defined via Dixmier trace. For a detailed account we refer to [2,9,10].

The Dixmier trace [15] is the unique extension of the usual trace to the class $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ which is an ideal in the algebra of bounded operators. The elements of this ideal are characterized by the condition that for any $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ the ordered eigenvalues λ_i of $|T|$ satisfy

$$\sup_N \frac{1}{\log N} \sum_{i=0}^{N} \lambda_i < \infty. \tag{5.1}$$

On this ideal the Dixmier trace $\mathrm{Tr}_\omega(\cdot)$ is defined as functional with the property

$$\mathrm{Tr}_\omega(T) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{i=0}^{N-1} \lambda_i. \tag{5.2}$$

If $\mathcal{A}$ is an arbitrary subalgebra of a C^*-algebra with a finitely summable K-cycle $(\mathcal{H}, D)$ then $|D|^{-d}$ is in $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ for some $d \in \mathbb{N}$, where d corresponds to the dimension of the underlying (non-commutative) space. Since

$$\mathrm{Tr}_\omega(|D|^{-d}) > 0, \tag{5.3}$$

an inner product on $\pi_D(\Omega\mathcal{A})$ is obtained by defining for each k

$$\begin{aligned} &(\cdot,\cdot)^k : \pi_D(\Omega^k\mathcal{A}) \times \pi_D(\Omega^k\mathcal{A}) \longrightarrow \mathbb{C}, \\ &(\pi_D(\omega_1), \pi_D(\omega_2))^k = \mathrm{Tr}_\omega(\pi_D(\omega_1^*)\pi_D(\omega_2)|D|^{-d}), \quad \omega_1, \omega_2 \in \Omega^k\mathcal{A}, \end{aligned} \tag{5.4}$$

which is positive if $\pi_D(\omega^*) = \pi_D(\omega)^*$, $\forall \omega \in \Omega\mathcal{A}$.

Let us denote by $\mathcal{H}_\pi^k$ the Hilbert space completion of $\pi_D(\Omega^k\mathcal{A})$ and let $P^{(k)}$ be the orthogonal projection of $\mathcal{H}_\pi^k$ onto the orthogonal complement of $\overline{\pi_D(\mathcal{J}^k)} \subset \mathcal{H}_\pi^k$ then an inner product on $\Omega_D\mathcal{A}$ can be defined for each k by

$$\begin{aligned} &\langle\cdot,\cdot\rangle^k : \Omega_D^k\mathcal{A} \times \Omega_D^k\mathcal{A} \longrightarrow \mathbb{C}, \\ &\langle\sigma_D(W_1), \sigma_D(W_2)\rangle^k = (P^{(k)}W_1, P^{(k)}W_2), \quad W_1, W_2 \in \pi_D(\Omega^k\mathcal{A}), \end{aligned} \tag{5.5}$$

which is positive if $(\cdot,\cdot)$ is positive.

This allows to identify $\Omega_D^k\mathcal{A}$ with a dense subspace of $\mathcal{H}_\pi^k$ and hence there is a map

$$c : \overline{\Omega_D^k\mathcal{A}} \longrightarrow \mathcal{H}_\pi^k \tag{5.6}$$

with $\mathrm{Im}(c) = \overline{\pi(\mathcal{J}^k)}^{\perp}$.

In the case, where $\mathcal{A} = C^\infty(\mathcal{M})$ is the algebra of smooth functions on a compact spin-manifold $\mathcal{M}$ and $D = \not\partial$ is the Dirac operator, $\Omega_D\mathcal{A}$ is the usual (complexified) de Rham algebra [2] and the inner product is

$$\langle w_1, w_2\rangle = \int_{\mathcal{M}} *\overline{w}_1 \wedge w_2, \quad w_1, w_2 \in \Omega_D^k\mathcal{A}, \tag{5.7}$$

where $*w_1$ is the Hodge dual of w_1.

Let us now turn to our case where the algebra is of the form $\mathcal{A} = C^\infty(I, \mathcal{A}_s)$ where we would like to introduce a substitution for integration on a space-like surface. However the "space-like" part of $\mathcal{A}$ and $\Omega_D\mathcal{A}$ is characterized by $\mathcal{A}_s$ and the smooth 1 parameter family of K-cycles $(\mathcal{H}_s, D_s)$ over $\mathcal{A}_s$, which are finitely summable by assumption. Therefore there is some d (the dimension of the "space-like" part of the non-commutative space) such that for any $t \in I$ $|D_s|^{-d}$ is an operator on $\mathcal{H}_s$ with $|D_s|^{-d} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_s)$ and

$$\mathrm{Tr}_\omega(|D_s|^{-d})_s > 0. \tag{5.8}$$

Here $\mathrm{Tr}_\omega(\cdot)_s$ denotes the Dixmier trace on $\mathcal{L}^{(1,\infty)}(\mathcal{H}_s)$. Since for any $t \in I$ any $W \in \pi_D(\Omega_D)$ is a bounded operator on $\mathcal{H}_s$ varying smoothly with t,

$$\begin{aligned}&(\cdot,\cdot)_s^k : \pi_D(\Omega^k\mathcal{A}) \times \pi_D(\Omega^k\mathcal{A}) \longrightarrow \mathbb{C}^\infty(I),\\ &(\pi_D(\omega_1), \pi_D(\omega_2))_s^k = \mathrm{Tr}_\omega(\pi_D(\omega_1^*)\pi_D(\omega_2)|D_s|^{-d})_s, \quad \omega_1, \omega_2 \in \Omega^k\mathcal{A}\end{aligned} \tag{5.9}$$

defines a positive inner product on $\pi_D(\Omega\mathcal{A})$ for any k and any (fixed) $t \in I$ if $\pi_D(\omega^*) = \pi_D(\omega)^*$, $\forall\omega \in \Omega\mathcal{A}$, a condition which is met in our case (see (vi)), i.e.

$$(W, W)_s = f(t) \geq 0, \quad \forall W \in \pi_D(\Omega\mathcal{A}),\ \forall t \in I. \tag{5.10}$$

With this inner product on $\pi_D(\Omega\mathcal{A})$ one can define an inner product on $\Omega_D\mathcal{A}$ as in the general construction. Let us denote by ${\mathcal{H}_s}_\pi^k$ the completion[4] of $\pi_D(\Omega^k\mathcal{A})$ with respect to $(\cdot,\cdot)_s$ and let $P_s^{(k)}$ be the orthogonal projection of ${\mathcal{H}_s}_\pi^k$ onto the orthogonal complement of $\pi_D(\mathcal{J}^k)$ then for each k

$$\begin{aligned}&\langle\cdot,\cdot\rangle_s^k : \Omega_D^k\mathcal{A} \times \Omega_D^k\mathcal{A} \longrightarrow \mathbb{C}^\infty(I),\\ &\langle\sigma_D(W_1), \sigma_D(W_2)\rangle_s^k = (P_s^{(k)}W_1, P_s^{(k)}W_2)_s, \quad W_1, W_2 \in \pi_D(\Omega^k\mathcal{A})\end{aligned} \tag{5.11}$$

defines a positive inner product on $\Omega_D\mathcal{A}$ for any $t \in I$. With this product we will define Lagrange functions and the Hamilton formalism. As in the general case there is a map

[4] We call a sequence convergent with respect to $(\cdot,\cdot)_s$ if it converges pointwise for all $t \in I$.

$$c_s : \overline{\Omega_D^k \mathcal{A}} \longrightarrow \mathcal{H}_{s\pi}^{\ k} \tag{5.12}$$

and hence we can identify $\Omega_D^k \mathcal{A}$ with a dense subspace of $\mathcal{H}_{s\pi}^{\ k}$.

The definition of the inner product in Eq. (5.11) allows for an important freedom in the choice of the K-cycle over $\mathcal{A}$, which deserves some discussion. For any $w_1 \in \Omega_{\mathcal{N}}^k \mathcal{A} \subset \Omega_D^k \mathcal{A}$ and any $w_2 \in \Omega_{\mathcal{N}}^{k-1} \mathcal{A} dt \subset \Omega_D^k \mathcal{A}$ it is

$$\langle w_1, w_2 \rangle_s = (c_s(w_1), c_s(w_2))_s = \mathrm{Tr}_\omega (c_s(w_1)^* c(w_2))_s = 0, \tag{5.13}$$

since $c_s(w_1)^* c(w_2)$ contains an odd number of commutators with D_s which are off diagonal (with respect to the block diagonal structure of Eq. (3.15)). This proves the following lemma.

Lemma 5. *The decomposition*

$$\Omega_D^k \mathcal{A} = \Omega_{\mathcal{N}}^k \mathcal{A} \oplus \Omega_{\mathcal{N}}^{k-1} \mathcal{A} dt \tag{5.14}$$

is orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_s$.

We have seen that condition (iii) of D, i.e. D has a compact inverse and $|D|^{-d} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, is crucial for the definition of the inner products (5.4) and (5.5). However, we will restrict ourselves to "integration on space-like" surfaces and hence use the inner products defined by Eqs. (5.9) and (5.11). Here we only need that D_s has a compact inverse and that $|D_d|^{-d} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_s)$ for some d which is guaranteed by condition (vi). We can use this freedom and change the definition of D_t by choosing γ^0 self-adjoint. As a consequence we find for any element $\omega \in \Omega_D \mathcal{A}$

$$(d_s \omega)^* = -d_s(\omega^*), \qquad (d_t \omega)^* = d_t(\omega^*). \tag{5.15}$$

Following Chamseddine et al. [16] we introduce a generalized metric on $\Omega_D^1 \mathcal{A}$.[5] In this context the $\mathcal{A}$-module $\Omega_D^1 \mathcal{A}$ is interpreted as the generalized cotangent bundle over a non-commutative space. We define the metric

$$g(\cdot,\cdot) : \Omega_D^1 \mathcal{A} \times \Omega_D^1 \mathcal{A} \longrightarrow \mathcal{A} \tag{5.16}$$

by the following equation

$$\langle A, g(v, w) \rangle_s = -\mathrm{Tr}_\omega (A^* c_s(v^*) c_s(w) \, |D_s|^{-d})_s, \quad \forall A \in \mathcal{A}; \quad v, w \in \Omega_D^1 \mathcal{A}. \tag{5.17}$$

This metric enjoys the property

$$g(vA, wB) = A^* g(v, w) B, \quad \forall A, B \in \mathcal{A}; \quad v, w \in \Omega_D^1 \mathcal{A}. \tag{5.18}$$

An important property of this metric is stated in the following theorem.

[5] Strictly speaking the metric is introduced on $\overline{\Omega_D^1 \mathcal{A}}$ which is the Hilbert space completion of $\Omega_D^1 \mathcal{A}$. However, we assume that the construction holds on $\Omega_D^1 \mathcal{A}$.

Theorem 2. *If γ^0, as defined in Eq.* (3.16), *is self-adjoint, i.e.*

$$\gamma_0 = \gamma_0^{\dagger} \tag{5.19}$$

then $g(\cdot,\cdot)$, as defined in Eq. (5.17), *is generalized Minkowskian metric, which is positive definite on $\Omega^1_{\mathcal{N}}\mathcal{A}$ and negative definite on the $\mathcal{A}$-module generated by dt.*

Proof. Applying the arguments presented in [16] to our case, we conclude that $g(\cdot,\cdot)$ defines a positive definite Riemannian metric on $\Omega^1_{\mathcal{N}}\mathcal{A} \subset \Omega^1_D\mathcal{A}$. From Lemma 5 we infer that

$$g(v, dt) = 0, \quad \forall v \in \Omega^1_{\mathcal{N}}\mathcal{A}. \tag{5.20}$$

From the definitions in Eq. (5.9), Eq. (5.11) and the definition of γ^0 in Eq. (3.16) it follows that

$$g(dt, dt) = -\gamma^0\gamma^0 = -N \in \mathcal{A} \tag{5.21}$$

and the theorem is proved. □

This theorem completely justifies the terminology of "space-like" and "time-like" since with the choice ${\gamma^0}^{\dagger} = \gamma^0$ it is possible to identify time like elements of $\Omega^1_D\mathcal{A}$ as elements with negative norm, i.e. elements $v \in \Omega^1_D\mathcal{A}$ with

$$g(v, v) = |v|^2 < 0. \tag{5.22}$$

For the rest of this article we will keep the choice ${\gamma^0}^{\dagger} = \gamma^0$, which means we are working on a non-commutative Minkowski space.

We end this section with some further definitions and some assumption on the algebra $\mathcal{A}$ which will be useful in the Hamiltonian framework. The first definition is a slight generalization of Eq. (5.17). We associate to any $v^{(l)} \in \Omega^l_D\mathcal{A},\ l \geq 0$ a map $i_l(v^{(l)})$, which is defined for all $k \geq 0$ by

$$\begin{aligned}\langle w_1, i_l(v^{(l)})w_2\rangle_s &= \mathrm{Tr}_\omega(\boldsymbol{c}_s(w_1^*)\boldsymbol{c}_s((v^{(l)})^*)\boldsymbol{c}_s(w_2)|D|^{-d})_s \\ &= \langle vw_1, w_2\rangle_s, \quad \forall w_1 \in \Omega^k_D\mathcal{A},\ \forall w_2 \in \Omega^{k+l}_D\mathcal{A}.\end{aligned} \tag{5.23}$$

This map is well defined as can be seen by applying the arguments presented in [16] for the definition of the metric. Thus we have defined a map which decreases the degree of forms

$$i_l(v^{(l)}) : \Omega^{k+l}_D\mathcal{A} \longrightarrow \overline{\Omega^k_D\mathcal{A}}. \tag{5.24}$$

For the second definition we have to make a further assumption on the algebra $\mathcal{A}$ and the K-cycle $(\mathcal{H}, D)$ over $\mathcal{A}$. Namely that for any $v \in \Omega^k_D\mathcal{A}, k > 0$, there is a $C_v \in \mathbb{R}$ such that for all $w \in \Omega^{k-1}_D\mathcal{A}$

$$|\langle v, dw\rangle_s|^2 \leq C_v\langle w, w\rangle_s. \tag{5.25}$$

This condition is fulfilled, for example, if $\mathcal{A} = C^\infty(M)$ and D is the Dirac-operator on M or if $\mathcal{A}$ is a finite dimensional algebra or if $\mathcal{A}$ is a product of the first two cases. Thus Eq. (5.25) is fulfilled for the class of algebras which has been used for model building in physics so far. This condition ensures that there is a well defined adjoint operator d_s^* of d_s on $\Omega_D^k \mathcal{A}$,

$$d_s^* : \Omega_D^k \mathcal{A} \longrightarrow \overline{\Omega_D^{k-1} \mathcal{A}}, \tag{5.26}$$

which is uniquely defined by

$$\langle d_s^* v, w \rangle_s := \langle v, d_s w \rangle_s, \quad \forall v \in \Omega_D^k \mathcal{A}, \ \forall w \in \Omega_D^{k-1} \mathcal{A}. \tag{5.27}$$

Furthermore we assume that the smooth 1-parameter family of K-cycles $(\mathcal{H}_s, D_s)$ is same [10], i.e.

$$\mathrm{Tr}_\omega([W_1, W_2]|D_s|^{-d}) = 0, \quad W_1, W_2 \in \pi_D(\Omega \mathcal{A}) \tag{5.28}$$

and

$$i_l(v^{(l)})(\Omega_D \mathcal{A}) \subset \Omega_D \mathcal{A}, \qquad d_s^*(\Omega_D \mathcal{A}) \subset \Omega_D \mathcal{A}. \tag{5.29}$$

These conditions are fulfilled in the above mentioned examples.

6. Lagrange and Hamilton function for Yang–Mills theory

Now we have all basic objects at hand which are necessary to define a Lagrange function and the corresponding Hamilton function for Yang–Mills theory in non-commutative geometry. However, we start with a brief exposition of Yang–Mills theory in non-commutative geometry as it was introduced by Connes and Lott [1,2], which allows us to fix our notation. A comprehensive presentation of this subject can be found in [2,9,10].

Yang–Mills theory is formulated on vector bundles. In the algebraic language a vector bundle is a finitely generated projective module over $\mathcal{A}$ which we denote by $\mathcal{E}$. Any finitely generated module $\mathcal{E}$ can be obtained from a free module $\mathcal{E}_0 = \mathcal{A}^N$ with the help of some idempotent $e \in \mathcal{A}^{N\times N}$, which means that we can write $\mathcal{E} = e\mathcal{A}^N$. In our case, the structure of $\mathcal{A} = C(I, \mathcal{A}_s)$ implies that $\mathcal{E} = C(I, \mathcal{E}_s)$, where $\mathcal{E}_s$ is a finitely generated projective module over $\mathcal{A}_s$.

Furthermore we need a Hermitian structure on $\mathcal{E}$, i.e. a sesquilinear form

$$(\cdot,\cdot)_\mathcal{E} : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{A} \tag{6.1}$$

with the following properties:
- $(A\zeta, B\eta)_\mathcal{E} = A^*(\zeta,\eta)_\mathcal{E} B, \ \forall \zeta,\eta \in \mathcal{E}, \ A, B \in \mathcal{A}$;
- $(\zeta,\zeta)_\mathcal{E} \geq 0, \ \forall \zeta \in \mathcal{E}$;
- $\mathcal{E}$ is self dual for $(\cdot,\cdot)_\mathcal{E}$.

If we write $\mathcal{E} = e\mathcal{A}^n$ the Hermitian structure requires that e is self-adjoint.

We extend $\mathcal{E}$ to a right module $\tilde{\mathcal{E}}$ over $\Omega_D \mathcal{A}$,

$$\tilde{\mathcal{E}}^k = \mathcal{E} \otimes_\mathcal{A} \Omega_D^k \mathcal{A}, \qquad \tilde{\mathcal{E}} = \mathcal{E} \otimes_\mathcal{A} \Omega_D \mathcal{A} \tag{6.2}$$

and also

$$(\cdot,\cdot)_{\tilde{\mathcal{E}}} : \tilde{\mathcal{E}} \times \tilde{\mathcal{E}} \longrightarrow \Omega_D\mathcal{A}. \tag{6.3}$$

A connection is defined as a linear map

$$\nabla : \tilde{\mathcal{E}}^k \longrightarrow \tilde{\mathcal{E}}^{k+1} \tag{6.4}$$

such that

$$\nabla(\zeta w) = \nabla(\zeta)w + (-1)^k dw, \quad \zeta \in \tilde{\mathcal{E}}^k, \ w \in \Omega_D\mathcal{A}. \tag{6.5}$$

One also requires that the connection is compatible with the metric $(\cdot,\cdot)_{\tilde{\mathcal{E}}}$, which for Euclidean K-cycles, i.e. for $D^\dagger = D$, is equivalent to the condition

$$(\zeta, \nabla\eta)_{\tilde{\mathcal{E}}} - (\nabla\zeta, \eta)_{\tilde{\mathcal{E}}} = d(\zeta,\eta)_{\tilde{\mathcal{E}}}, \quad \zeta,\eta \in \mathcal{E}. \tag{6.6}$$

The set of compatible connections form an affine space and for any two compatible connections ∇, ∇' we have

$$\nabla - \nabla' = \boldsymbol{A} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D\mathcal{A}). \tag{6.7}$$

Note, that the definition of a compatible connection depends on the definition of the $*$-operation on $\Omega\mathcal{A}$ and the choice of D for the K-cycle over $\mathcal{A}$. In our case we have $D_s^\dagger = D_s$ and $D_t^\dagger = -D_t$. Thus condition (6.6) is valid only on the space-like part of the connection. For the time-like part of the connection the compatibility condition reads

$$(\zeta, \nabla_t\eta)_{\tilde{\mathcal{E}}} + (\nabla_t\zeta, \eta)_{\tilde{\mathcal{E}}} = d_t(\zeta,\eta)_{\tilde{\mathcal{E}}}, \quad \zeta,\eta \in \mathcal{E}. \tag{6.8}$$

One can check (see, e.g. [10]) that for $\mathcal{E} = e\mathcal{A}^N$

$$\nabla_0\zeta = ed\zeta, \quad \zeta \in \mathcal{E} \tag{6.9}$$

defines a compatible connection. Thus any compatible connection ∇ can be written as

$$\nabla = \nabla_0 + \boldsymbol{A}, \quad \boldsymbol{A} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D\mathcal{A}). \tag{6.10}$$

Here we used that the restriction on ∇ to $\mathcal{E}$ already defines the connection uniquely on $\tilde{\mathcal{E}}$. The curvature $\boldsymbol{F}$ is obtained by taking the square of the connection

$$\boldsymbol{F} = \nabla^2 = e(de)^2 + edee\alpha e + ed\alpha e - e\alpha de + e\alpha e\alpha \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D\mathcal{A}) \tag{6.11}$$

with $e\alpha e = \boldsymbol{A}$ and $\alpha \in \mathcal{A}^{N\times N} \otimes_{\mathcal{A}} \Omega^1_D\mathcal{A}$.

Connection and curvature transform covariantly under unitary transformations $U(\mathcal{E}) = \{u \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \mid uu^* = u^*u = 1\}$, i.e.

$$\boldsymbol{F}' = u\boldsymbol{F}u^*, \qquad \nabla' = u\nabla u^*, \tag{6.12}$$

from which we infer that the vector-potential $\boldsymbol{A}$ transforms as follows:

$$\boldsymbol{A}' = u\boldsymbol{A}u^* + udu^*. \tag{6.13}$$

The inner product on $\Omega_D \mathcal{A}$ and the Hermitian structure on $\mathcal{E}$ induce a natural inner product on $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D \mathcal{A})$ for any k. We want to construct this product explicitly and therefore we note that any $\boldsymbol{T} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D \mathcal{A})$ can be written as

$$\boldsymbol{T} = \sum_{r,s=1}^{N} e_{ik} w_{rs} e_{lj}, \quad w_{kl} \in \Omega^k_D \mathcal{A}. \tag{6.14}$$

In this notation the inner product $(\cdot,\cdot)$ on $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D \mathcal{A})$ can be defined as

$$(\cdot,\cdot) : \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D \mathcal{A}) \times \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D \mathcal{A}) \longrightarrow C^{\infty}(I), \tag{6.15}$$

$$\begin{aligned}(\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}) &= \mathrm{tr}\,\mathrm{Tr}_{\omega}(\boldsymbol{c}_s(\boldsymbol{T}^{(1)\dagger})\boldsymbol{c}_s(\boldsymbol{T}^{(2)})|D_s|^{-d})_s \\ &= \sum_{j,k=1}^{N} \sum_{r,s=1}^{N} \sum_{p,q=1}^{N} \langle e_{rj} w^{(1)}_{sr} e_{ks}, e_{jp} w^{(2)}_{pq} e_{qk} \rangle_s, \quad w^{(1)}_{rs}, w^{(2)}_{pq} \in \Omega^k_D \mathcal{A}.\end{aligned} \tag{6.16}$$

We use this inner product to define the Lagrange function L for Yang–Mills theory in non-commutative geometry,

$$L(\boldsymbol{A}) = -\tfrac{1}{4}(\boldsymbol{F}, \boldsymbol{F}) \in C^{\infty}(I). \tag{6.17}$$

The action S for Yang–Mills theory is obtained by integrating the Lagrange function L over time

$$S(\boldsymbol{A}) = \int_{t_1}^{t_2} \mathrm{d}t\, L(\boldsymbol{A}) = -\frac{1}{4} \int_{t_1}^{t_2} \mathrm{d}t\, (\boldsymbol{T}, \boldsymbol{T}). \tag{6.18}$$

So far we have discussed the general case where $\mathcal{E}$ is a finitely generated $\mathcal{A}$-module. From now on we will restrict ourselves to the case where $\mathcal{E} = \mathcal{A}^N$ is a free module. However, note that the formalism which will be presented in the following can be generalized to finitely generated $\mathcal{A}$-modules. The reason for the restriction is just to avoid unnecessarily complicated formulas.

Because of Lemma 5 there is also an orthogonal decomposition of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D \mathcal{A})$ with respect to the inner product $(\cdot,\cdot)$:

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D \mathcal{A}) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_N \mathcal{A}) \oplus \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^{k-1}_N \mathcal{A} dt) \tag{6.19}$$

and therefore we can decompose $\boldsymbol{F}$ as follows:

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{F}_{st} + \boldsymbol{B}, \\ \boldsymbol{F}_{st} &= d_t \boldsymbol{A}_s + \nabla_s \boldsymbol{A}_t \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_N \mathcal{A} dt), \\ \boldsymbol{B} &= d_s \boldsymbol{A}_s + \boldsymbol{A}_s^2 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_N \mathcal{A}),\end{aligned} \tag{6.20}$$

where $A_s \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}}\mathcal{A})$ is the space-like part of A and $A_t \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}dt)$ is the time-like part of A. $\nabla_s = d_s + A_s$ denotes the space-like part of the connection. With this decomposition L becomes

$$L = -\tfrac{1}{4}((F_{st}, F_{st}) + (B, B)), \tag{6.21}$$

where the first term on the right hand side is positive and the second term is negative.

Now we define the canonical momenta in the usual way, namely the variation of L with respect to the time derivative of the variables at some fixed time t. In our case we have to vary L with respect to $d_t A$. We find that

$$E_s = \frac{\delta L}{\delta dt A_s} = -\frac{1}{2}(F_{st}, \cdot) \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}}\mathcal{A}dt)^{\star}_t, \tag{6.22}$$

$$E_t = \frac{\delta L}{\delta dt A_t} = 0. \tag{6.23}$$

Here $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}}\mathcal{A}dt)^{\star}_t$ denotes the image of the map (at some fixed time t),

$$\star : \mathcal{T}_{st} \longrightarrow \mathcal{T}^{\star}_{st}, \qquad \star(T) = (T, \cdot), \quad T \in \mathcal{T}, \tag{6.24}$$

restricted to $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}}\mathcal{A}dt)$, where $\mathcal{T}_{st}$ is the Hilbert space completion of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}}\mathcal{A}dt)$ and $\mathcal{T}^{\star}_{st}$ is the dual Hilbert space of $\mathcal{T}_{st}$. However, we use the map $\star^{-1}$ to identify $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}}\mathcal{A}dt)^{\star}_t$ with $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}}\mathcal{A}dt)_t$ and thus we consider the canonical momentum $\mathcal{E}$ as an element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}}\mathcal{A}dt)_t$. As in usual Yang–Mills theory we see that there are no canonical momenta for A_t. Thus Eqs. (6.23) are primary constraints.

We define the Hamiltonian H as

$$\begin{aligned} H &= E(d_t A) - L = \tfrac{1}{4}(-(E_s, E_s) + (B, B)) - (\nabla^{\star}_s E_s, A_t) \\ &= H_0 - G(A_t), \end{aligned} \tag{6.25}$$

$$H_0 = \tfrac{1}{4}(-(E_s, E_s) + (B, B)), \qquad G = (\nabla^{\star}_s E_s, A_t), \tag{6.26}$$

where

$$\nabla^{*}_s : \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D\mathcal{A}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^{k-1}_D\mathcal{A}) \tag{6.27}$$

is defined by

$$\begin{aligned} &(T_1, \nabla_s T_2) = (\nabla^{\star}_s T_1, T_2), \\ &T_1 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D\mathcal{A}), \quad T_2 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^{k-1}_D\mathcal{A}). \end{aligned} \tag{6.28}$$

Such a map exists because of the assumption (5.29). Note that H_0 is positive since it is $(E_s, E_s) \le 0$. As one may have expected, the Hamiltonian for Yang–Mills theory in non-commutative geometry is formally exactly the same as for conventional Yang–Mills theory. However, the Hamiltonian in Eq. (6.25) is defined purely algebraic and therefore still makes sense in cases where there is no space–time manifold.

7. The Poisson bracket and time evolution

From the discussion of the previous section we infer that the canonical phase-space Γ_0 of Yang–Mills theory in non-commutative geometry is

$$\Gamma_0 \subset \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}} \mathcal{A} dt)_t \oplus \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}} \mathcal{A})_t, \tag{7.1}$$

where the subscript t indicates that we have fixed the time t when the momenta were defined. Thus the elements of the phase-space Γ_0 do not have any time dependence. More generally, we define for any k

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D \mathcal{A})_t = \frac{\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D \mathcal{A})}{\mathcal{I}^k_t}, \tag{7.2}$$

where $\mathcal{I}_t$ is the graded ideal,

$$\mathcal{I}_t = \{z \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D \mathcal{A}) \,|\, z(t) = 0\}. \tag{7.3}$$

Since from now on all objects are considered at some fixed time t we drop the subscript t in order to simplify notation.

However, there are some restrictions on the elements of Γ_0. The first one is a reality constraint on the variables which originates from the condition that $\boldsymbol{A}$ is a compatible connection, i.e.

$$\boldsymbol{A}^\dagger = \boldsymbol{A}. \tag{7.4}$$

Since

$$\boldsymbol{E} = -d_t \boldsymbol{A} - \nabla_s \boldsymbol{A}_t, \tag{7.5}$$

the compatibility condition on $\boldsymbol{A}$ implies that

$$\boldsymbol{E}^\dagger = -\boldsymbol{E}. \tag{7.6}$$

Thus the canonical phase-space of Yang–Mills theory in non-commutative geometry is

$$\Gamma_0 = \{(\boldsymbol{A}, \boldsymbol{E}) \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}} \mathcal{A} dt)_t \oplus \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{N}} \mathcal{A})_t \,|\, (\boldsymbol{A}^\dagger, \boldsymbol{E}^\dagger) = (\boldsymbol{A}, -\boldsymbol{E})\}. \tag{7.7}$$

In Eq. (7.7) we also have used the fact that there is no canonical momentum for $\boldsymbol{A}_t$ and hence this variable plays the role of a Lagrange multiplier. Thus we can read off from Eq. (6.25) the secondary constraint on the elements $\boldsymbol{A}$, $\boldsymbol{E}$ of Γ_0 (we suppressed the index s), namely

$$\boldsymbol{G}(\boldsymbol{A}_t) = (\nabla^* \boldsymbol{E}, \boldsymbol{A}_t) = 0, \quad \forall \boldsymbol{A}_t \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{A} dt). \tag{7.8}$$

This is the Gauß-law in non-commutative geometry.

However, we have not defined a Poisson bracket for this space so far. A Poisson bracket is a antisymmetric linear map $\{\cdot, \cdot\}$ on a suitable space of functions $\mathcal{C}$ on Γ_0. Therefore we first have to define $\mathcal{C}$.

We take for $\mathcal{C}$ the algebra of functions on Γ_0 which contain arbitrary finite powers of the elements $\boldsymbol{A}, \boldsymbol{E} \in \Gamma_0$ and their derivatives (of finite order). For any $w \in \Omega_D\mathcal{A}$ we define

$$\begin{aligned} w^{(2k,0)}w &:= (d_s^* d_s)^k w, & w^{(0,2k)}w &:= (d_s d_s^*)^k w, \\ w^{(2k+1,0)}w &:= (d_s d_s^*)^k d_s w, & w^{(0,2k+1)}w &:= (d_s^* d_s)^k d_s^* w. \end{aligned} \tag{7.9}$$

General combinations of derivatives are denoted by $w^{(k,l)} = w^{(k,0)} + w^{(0,l)}$, $w \in \Omega_D\mathcal{A}$. Those elements are well defined because of assumption (5.29).

Furthermore we need the analogue of partial integration in non-commutative geometry. For this purpose we define for any $k \geq 0$ the map pr_k,

$$pr_k : \pi_D(\Omega\mathcal{A}) \longrightarrow \Omega_D^k\mathcal{A}, \tag{7.10}$$

by the equation

$$\langle v, pr_k(W)\rangle_s = \mathrm{Tr}_\omega(\boldsymbol{c}_s(v)^* W|D_s|^{-d})_s, \quad \forall v \in \Omega_D^k\mathcal{A},\ W \in \pi_D(\Omega\mathcal{A}). \tag{7.11}$$

Again assumption (5.29) ensures that this map exists. With the help of this map we can define the analogue of partial integration for all $v \in \Omega_D^k\mathcal{A}$, $W \in \boldsymbol{c}_s(\Omega_D\mathcal{A})$ by

$$\begin{aligned} \mathrm{Tr}_\omega(\boldsymbol{c}_s(d_s v)W|D_s|^{-d})_s &= -\mathrm{Tr}_\omega(\boldsymbol{c}_s(v)\boldsymbol{c}_s(d_s^* pr_{k+1}(W))|D_s|^{-d})_s, \\ \mathrm{Tr}_\omega(\boldsymbol{c}_s(d_s^* v)W|D_s|^{-d})_s &= -\mathrm{Tr}_\omega(\boldsymbol{c}_s(v)\boldsymbol{c}_s(d_s pr_{k-1}(W))|D_s|^{-d})_s. \end{aligned} \tag{7.12}$$

It is convenient to consider the subalgebra $\mathcal{P}(\Gamma_0)$ of the algebra of continuous maps from Γ_0 to $\mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E} \otimes_\mathcal{A} \pi_D(\Omega\mathcal{A}))$, which is generated by elements $P_m^{(j,k)}$ of the form

$$P_m^{(j,k)} = \boldsymbol{c}_s(pr_m(\boldsymbol{c}_s(z_1)\cdots\boldsymbol{c}_s(z_n))^{(j,k)}), \quad j,k \geq 0,\ n > 0 \tag{7.13}$$

with

$$z_l \in \{\boldsymbol{A}, \boldsymbol{E}, N_0, N_s, N_t\}, \quad (\boldsymbol{A}, \boldsymbol{E}) \in \Gamma_0. \tag{7.14}$$

The elements N, N_s, N_t with

$$\begin{aligned} N &\in \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E}), & N^\dagger &= -N; \\ N_s &\in \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E} \otimes_\mathcal{A} \Omega_\mathcal{N}^1\mathcal{A}), & N_s^\dagger &= N_s; \\ N_t &\in \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E} \otimes_\mathcal{A} \mathcal{A}dt), & N_t^\dagger &= -N_t \end{aligned} \tag{7.15}$$

play the role of test functions. We obtain $\mathcal{C} \subset C(\Gamma_0, \mathbb{C})$ by taking the trace of the elements in $\mathcal{P}$

$$\mathcal{C} := \{F \in C(\Gamma_0, \mathbb{C}) \mid F = \mathrm{tr}\,\mathrm{Tr}_\omega(P|D_s|^{-d})_s, P \in \mathcal{P}\}. \tag{7.16}$$

Having specified the space of functions on Γ_0 we define the Poisson bracket $\{\cdot,\cdot\}$ with the help of a functional $G(\cdot,\cdot)$

$$\begin{aligned} \{\mathrm{tr}\,\mathrm{Tr}_\omega(P_1|D_s|^{-d})_s, \mathrm{tr}\,\mathrm{Tr}_\omega(P_2|D_s|^{-d})_s\} &= \mathrm{tr}\,\mathrm{Tr}_\omega(G(P_1,P_2)|D_s|^{-d})_s \\ &= -\mathrm{tr}\,\mathrm{Tr}_\omega(G(P_2,P_1)|D_s|^{-d})_s, \end{aligned} \tag{7.17}$$

The functional $G(\cdot,\cdot)$ is the non-commutative generalization of the δ-distribution which is defined by the following rules:

For any $P_p, P'_q \in \mathcal{P},\ 1 \le p \le k,\ 1 \le q \le l$ it is

$$\begin{aligned}
&\operatorname{tr}\operatorname{Tr}_\omega(G(P_1 \cdots P_k, P'_1 \cdots P'_l)|D_s|^{-d})_s \\
&\quad = \sum_{cp_k}\sum_{cp_l} \operatorname{tr}\operatorname{Tr}_\omega(P_{cp_k(1)} \cdots P_{cp_k(k-1)} \\
&\qquad\quad \times G(P_{cp_k(k)}, P'_{cp_l(1)})P'_{cp_l(2)} \cdots P'_{cp_l(l)}|D_s|^{-d})_s,
\end{aligned} \tag{7.18}$$

where $\sum_{cp_k}$ denotes the sum over the cyclic permutations of the first k indices and $\sum_{cp_l}$ denotes the sum over the cyclic permutations of the last l indices.

For any $\boldsymbol{c}_s(d_s v) \in \mathcal{P},\ v \in \Omega_D^k \mathcal{A}$ and for any $\boldsymbol{c}_s(d_s^* v) \in \mathcal{P},\ v \in \Omega_D^k \mathcal{A}$ it is $\forall P_1, P_2 \in \mathcal{P}$

$$\begin{aligned}
&\operatorname{tr}\operatorname{Tr}_\omega(G(P_1, \boldsymbol{c}_s(d_s v))P_2|D_s|^{-d})_s \\
&\quad = -\operatorname{tr}\operatorname{Tr}_\omega(G(P_1, \boldsymbol{c}_s(v))\boldsymbol{c}_s(d_s^* pr_{k+1}(P_2))|D_s|^{-d})_s, \\
&\operatorname{tr}\operatorname{Tr}_\omega(G(P_1, \boldsymbol{c}_s(d_s^* v))P_2|D_s|^{-d})_s \\
&\quad = -\operatorname{tr}\operatorname{Tr}_\omega(G(P_1, \boldsymbol{c}_s(v))\boldsymbol{c}_s(d_s pr_{k-1}(P_2))|D_s|^{-d})_s.
\end{aligned} \tag{7.19}$$

And finally we define for the basic fields $z_1, z_2 \in \{\boldsymbol{A}, \boldsymbol{E},\ N_0, N_s, N_t\}$

$$\begin{aligned}
&\operatorname{tr}\operatorname{Tr}_\omega(P_1 G(\boldsymbol{c}_s(z_1), \boldsymbol{c}_s(z_2))P_2|D_s|^{-d})_s \\
&\quad = \begin{cases}
\operatorname{tr}\operatorname{Tr}_\omega(P_1 {\gamma^0}^{-1}\mathrm{id}_{\mathcal{E}} P_2|D_s|^{-d})_s & \text{if } z_1 = \boldsymbol{A}, z_2 = \boldsymbol{E} \\
-\operatorname{tr}\operatorname{Tr}_\omega(P_1 {\gamma^0}^{-1}\mathrm{id}_{\mathcal{E}} P_2|D_s|^{-d})_s & \text{if } z_1 = \boldsymbol{E}, z_2 = \boldsymbol{A} \\
0 & \text{otherwise.}
\end{cases}
\end{aligned} \tag{7.20}$$

This completes the definition of the phase-space and the Poisson algebra.

The time evolution of the system is determined by the Hamiltonian H. For any element $F \in \mathcal{C}$ it is

$$\dot{F} = \{F, H\}, \tag{7.21}$$

where the dot denotes the time derivative of F. However, the Hamiltonian is not uniquely defined for this system since for some arbitrary $\Lambda \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E})$ we can add $\boldsymbol{G}(\Lambda dt)$ to the Hamiltonian without changing physics. This is possible because $\boldsymbol{G}(\Lambda dt)$ has to vanish on the physical subspace of Γ_0. Furthermore, consistency requires that the condition (7.8) is time-independent which leads to the following equations:

$$\{\boldsymbol{G}(\Lambda dt), H_0\} \approx 0\,, \tag{7.22}$$

$$\{\boldsymbol{G}(\Lambda_1 dt), \boldsymbol{G}(\Lambda_2 dt)\} \approx 0. \tag{7.23}$$

Here $\approx$ means that the equations hold modulo constraints. This implies that the constraints have to form a closed algebra.

Let us check that Eqs. (7.22, 7.23) are satisfied. We start with Eq. (7.23):

$$\begin{aligned}
\{\boldsymbol{G}(\Lambda_1 dt), \boldsymbol{G}(\Lambda_2 dt)\} &= \operatorname{tr}\operatorname{Tr}_\omega(G(\boldsymbol{c}_s(\nabla_s \Lambda_1)\gamma^0 \boldsymbol{c}_s(\boldsymbol{E}), \boldsymbol{c}_s(\nabla_s \Lambda_2)\gamma^0 \boldsymbol{c}_s(\boldsymbol{E}))|D_s|^{-d})_s \\
&= -\operatorname{tr}\operatorname{Tr}_\omega(\boldsymbol{c}_s(\nabla_s \Lambda_1)(\Lambda_2 \gamma^0 \boldsymbol{c}_s(\boldsymbol{E}) - \gamma^0 \boldsymbol{c}_s(\boldsymbol{E})\Lambda_2)|D_s|^{-d})_s \\
&\quad + \operatorname{tr}\operatorname{Tr}_\omega((\Lambda_1 \gamma^0 \boldsymbol{c}_s(\boldsymbol{E}) - \gamma^0 \boldsymbol{c}_s(\boldsymbol{E})\Lambda_1)\boldsymbol{c}_s(\nabla_s \Lambda_2)|D_s|^{-d})_s \\
&= -\boldsymbol{G}((\Lambda_1 \Lambda_2 - \Lambda_2 \Lambda_1)dt).
\end{aligned} \tag{7.24}$$

Before we turn to Eq. (7.22) it is useful to compute the following bracket:

$$\begin{aligned}&\operatorname{tr}\operatorname{Tr}_\omega(\Lambda\gamma^0 G(c_s(\boldsymbol{E}),\tfrac{1}{2}c_s(\boldsymbol{B})^2)|D_s|^{-d})_s\\&\quad=-\operatorname{tr}\operatorname{Tr}_\omega(\Lambda c_s(d_s^*\boldsymbol{B})|D_s|^{-d})_s+\operatorname{tr}\operatorname{Tr}_\omega((c_s(\boldsymbol{A})c_s(\boldsymbol{B})-c_s(\boldsymbol{B})c_s(\boldsymbol{A}))|D_s|^{-d})_s\\&\quad=-\operatorname{tr}\operatorname{Tr}_\omega(\Lambda c_s(\nabla_s^*\boldsymbol{B})|D_s|^{-d})_s.\end{aligned}\tag{7.25}$$

If we now insert $\Lambda = c_s(\nabla_s \Lambda_0)$ in Eq. (7.25) we obtain

$$\{G(\Lambda_0 dt),\tfrac{1}{2}\operatorname{tr}\operatorname{Tr}_\omega(c_s(\boldsymbol{B})^2|D_s|^{-d})_s\}=-\operatorname{tr}\operatorname{Tr}_\omega(c_s(\nabla_s\Lambda)c_s(\nabla_s^*\boldsymbol{B})|D_s|^{-d})_s=0.\tag{7.26}$$

The remaining part is

$$\{G(\Lambda_0 dt),\tfrac{1}{2}\operatorname{tr}\operatorname{Tr}_\omega(c_s(\boldsymbol{E})^2|D_s|^{-d})_s\}=\operatorname{tr}\operatorname{Tr}_\omega(\Lambda_0 c_s(\boldsymbol{E})^2-c_s(\boldsymbol{E})^2\Lambda_0)|D_s|^{-d})_s=0.\tag{7.27}$$

Hence the conditions (7.22) and (7.23) are fulfilled and the constraints $G(\Lambda dt)$ form a complete set of first-class constraints generating the symmetry of the theory. Thus the observables of the theory are elements $F \in C$ with

$$\{G(\Lambda dt), F\} = 0.\tag{7.28}$$

The time evolution of the basic fields $\boldsymbol{A}, \boldsymbol{E}$ can be computed by considering

$$\{\operatorname{tr}\operatorname{Tr}_\omega(c_s(\Lambda)c_s(\boldsymbol{A})|D_s|^{-d})_s, H_0\}=\operatorname{tr}\operatorname{Tr}_\omega(c_s(\Lambda)(\gamma^0)^{-1}c_s(\boldsymbol{E})|D_s|^{-d})_s.\tag{7.29}$$

From this and Eq. (7.25) we infer that the time evolution of the basic fields is (modulo gauge transformations)

$$\dot{\boldsymbol{A}}=-pr_1((\gamma^0)^{-1}c_s(\boldsymbol{E})),\qquad \dot{\boldsymbol{E}}=-pr_2(\gamma^0)^{-1}\nabla_s^*\boldsymbol{B}.\tag{7.30}$$

Equivalently, with $\boldsymbol{E} = \boldsymbol{E}_0 dt$, we can write

$$\dot{\boldsymbol{A}}=\boldsymbol{E}_0,\qquad \dot{\boldsymbol{E}}_0=-N^{-\frac{1}{2}}\nabla_s^*\boldsymbol{B}.\tag{7.31}$$

8. Examples

In this section we apply the general construction, presented in the previous sections, to two examples, which are, more or less, standard (toy) examples in non-commutative geometry applied to elementary particle physics. In the first one the algebra $\mathcal{A}_s$ is a sum of two identical finite dimensional algebras of complex $n \times n$ matrices. This is basically the setting of the "Two-Point Space" as it was presented in [2]. The "Yang–Mills Theory" on this discrete space generates a Higgs potential and spontaneous symmetry breaking.

In the second example the algebra of the first example is enlarged by the algebra of smooth functions on a compact Riemannian manifold. This leads to a gauge theory with conventional gauge bosons and Higgs bosons. The gauge symmetry of the model is $U(n) \times U(n)$ which is broken to $U(n)$. One might interpret this example as a model with a left–right chiral symmetry which is broken spontaneously to a vector-like symmetry. However, since we do not yet have fermions included in our construction, such an interpretation might be a little bit artificial.

8.1. The two-point space

We start with the discrete space and take for $\mathcal{A}_s$,

$$\mathcal{A}_s = \mathbb{C}^{n\times n} \oplus \mathbb{C}^{n\times n}, \tag{8.1}$$

which represents the space-like part of the algebra $\mathcal{A}$ in this example. A general discussion of Connes' generalized differential algebra constructed out of matrix-algebras can be found, e.g., in[17].

The complete algebra $\mathcal{A}$ over space-time is then

$$\mathcal{A} = C^\infty(\mathbb{R}, \mathbb{C}^{n\times n} \oplus \mathbb{C}^{n\times n}). \tag{8.2}$$

The Hilbert space $\mathcal{H}_s$ is

$$\mathcal{H}_s = (\mathbb{C}^n \oplus \mathbb{C}^n) \otimes \mathbb{C}^G \otimes \mathbb{C}^2, \tag{8.3}$$

where $\mathbb{C}^G$ denotes the "generation-space" with $G > 1$ and the $\mathbb{C}^2$ factor is needed for the construction on γ^0. The representation π_s is given for all $A = (A_1, A_2) \in \mathcal{A}_s$ as

$$\pi_s(A) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \otimes 1_{\mathbb{C}^G} \otimes 1_{\mathbb{C}^2}. \tag{8.4}$$

We take for the space-like operator D_s

$$D_s = \begin{pmatrix} 0 & \tilde{D}_s \\ \tilde{D}_s & 0 \end{pmatrix}, \qquad \tilde{D}_s = \begin{pmatrix} 0 & \mu \\ \mu^\dagger & 0 \end{pmatrix} \otimes M, \tag{8.5}$$

where $M \in \mathbb{C}^{G\times G}$, $M^2 \neq \alpha 1_{\mathbb{C}^G}$, $M^2 \neq 0$ is a matrix in generation space which guarantees that the representation of two-forms on $\mathcal{H}_s$ is linear independent from the representation of $\mathcal{A}_s$. We choose $\mu \in \mathbb{C}^{n\times n}$ such that $\mu\mu^\dagger = \mu^\dagger\mu = \lambda^2 1_{\mathbb{C}^n}$. Thus the space-like K-cycle $(\mathcal{H}_s, D_s)$ over $\mathcal{A}_s$ is defined and the extension to a K-cycle $(\mathcal{H}, D)$ over $\mathcal{A}$ along the lines described in Section 3 is straightforward:

$$\mathcal{H} = L_2(\mathbb{R}, (\mathbb{C}^n \oplus \mathbb{C}^n) \otimes \mathbb{C}^G \otimes \mathbb{C}^2),$$
$$D = D_t + D_s, \quad D_t = \begin{pmatrix} 1\partial_t & 0 \\ 0 & -1\partial_t \end{pmatrix}, \tag{8.6}$$

where the 1 in the definition of D_t refers to the unit in $\mathbb{C}^{2n} \otimes \mathbb{C}^G$. The representation π maps elements of $\mathcal{A}$ onto time-dependent blockdiagonal elements of the same form as in Eq. (8.4). The remaining element in the general set-up which we have to specify is the $\mathcal{A}$-module $\mathcal{E}$. We take the simplest choice, i.e. $\mathcal{E} = \mathcal{A}$.

Now we can write down the connection one form $\boldsymbol{A} = \boldsymbol{A}_t + \boldsymbol{A}_s$:

$$\boldsymbol{A}_t = \begin{pmatrix} \boldsymbol{A}_1 & 0 \\ 0 & \boldsymbol{A}_2 \end{pmatrix}, \qquad \boldsymbol{A}_s = \begin{pmatrix} 0 & \phi \\ \phi^\dagger & 0 \end{pmatrix}. \tag{8.7}$$

$\boldsymbol{A}_1, \boldsymbol{A}_2$ are anti-Hermitian $n \times n$ matrices multiplied by dt and ϕ is a complex $n \times n$ matrix of (matrix-) form degree 1, i.e. it is a $n \times n$ matrix multiplied by M.

The curvature $\boldsymbol{F} = \boldsymbol{F}_{st} + \boldsymbol{B}$ of $\boldsymbol{A}$ is given by

$$\boldsymbol{F}_{st} = \begin{pmatrix} 0 & \dot{\phi} dt + \boldsymbol{A}_1(\mu + \phi) + (\mu + \phi)\boldsymbol{A}_2 \\ \dot{\phi}^\dagger dt + (\mu^\dagger + \phi^\dagger)\boldsymbol{A}_1 + \boldsymbol{A}_2(\mu^\dagger + \phi^\dagger) & 0 \end{pmatrix},$$

$$\boldsymbol{B} = \begin{pmatrix} \phi\mu^\dagger + \mu\phi^\dagger + \phi\phi^\dagger & 0 \\ 0 & \phi^\dagger\mu + \mu^\dagger\phi + \phi^\dagger\phi \end{pmatrix}. \tag{8.8}$$

Since the space-like part $\mathcal{A}_s$ of the algebra $\mathcal{A}$ is finite dimensional the Dixmier-trace in the definition of the Lagrange function reduces to the normal trace and hence the Lagrange function L is

$$\begin{aligned} L &= -\tfrac{1}{4}\mathrm{tr}(F^\dagger F) \\ &= \tfrac{1}{2}\mathrm{tr}[(\dot{\phi}\gamma^0 + \boldsymbol{A}_1(\mu + \phi) + (\mu + \phi)\boldsymbol{A}_2) \\ &\qquad \times(\dot{\phi}^\dagger\gamma^0 + (\mu^\dagger + \phi^\dagger)\boldsymbol{A}_1 + \boldsymbol{A}_2(\mu^\dagger + \phi^\dagger))] - V(\phi) \end{aligned} \tag{8.9}$$

with

$$V(\phi) = \tfrac{1}{4}\mathrm{tr}[(\phi\mu^\dagger + \mu\phi^\dagger + \phi\phi^\dagger)(\phi^\dagger\mu + \mu^\dagger\phi + \phi^\dagger\phi)]. \tag{8.10}$$

Now we turn to the Hamilton formalism and find for the momentum $\boldsymbol{E}$,

$$\boldsymbol{E} = \begin{pmatrix} 0 & -\pi^\dagger \\ \pi & 0 \end{pmatrix}, \tag{8.11}$$

with

$$\pi = \dot{\phi}^\dagger dt + (\mu^\dagger + \phi^\dagger)\boldsymbol{A}_1 + \boldsymbol{A}_2(\mu^\dagger + \phi^\dagger) . \tag{8.12}$$

Thus the Hamiltonian $H = H_0 - G(\boldsymbol{A}_t)$ is given by

$$H_0 = \mathrm{tr}(\pi^\dagger\pi) + V(\phi), \tag{8.13}$$

$$G(\boldsymbol{A}_t) = \mathrm{tr}[\boldsymbol{E}(D_s + \boldsymbol{A}_s)\boldsymbol{A}_t + \boldsymbol{E}\boldsymbol{A}_t(D_s + \boldsymbol{A}_s)]. \tag{8.14}$$

The Gauß-law constraints

$$G((\Lambda_1, \Lambda_2)dt) = 0, \quad (\Lambda_1, \Lambda_2) = \Lambda \in \mathcal{A}_s, \ \Lambda^\dagger = -\Lambda \tag{8.15}$$

generate the Lie-algebra of the $U(n) \times U(n)$ symmetry group. The phase-space variables transform as follows:

$$\delta\pi = \Lambda_2\pi - \pi\Lambda_1, \qquad \delta\phi = \Lambda_1(\phi + \mu) - (\phi + \mu)\Lambda_2. \tag{8.16}$$

The inhomogeneous transformation property of ϕ is due to the fact that ϕ is part of the connection in this formalism. However, a substitution

$$\varphi = \phi + \mu \tag{8.17}$$

leads to a homogeneous transformation property

$$\delta\varphi = \Lambda_1\varphi - \varphi\Lambda_2. \tag{8.18}$$

The potential V reads in this new variable

$$V(\varphi) = \tfrac{1}{4}\mathrm{tr}(\varphi\varphi^\dagger - \lambda^2)(\varphi^\dagger\varphi - \lambda^2). \tag{8.19}$$

For the time-evolution of the system we find

$$\dot\varphi = \pi^\dagger, \qquad \dot\pi = -\tfrac{1}{4}[\varphi^\dagger(\varphi^\dagger\varphi - \lambda^2) + (\varphi\varphi^\dagger - \lambda^2)\varphi^\dagger]. \tag{8.20}$$

We see that there are two configurations in phase-space, which are stable under time evolution. The first one is $\pi = 0, \varphi = 0$, which is metastable and $\pi = 0, \varphi^\dagger\varphi = \lambda^2$ which is stable. The second configuration is the vacuum expectation value of the Higgs-field. By choosing for the vacuum expectation value φ_0,

$$\varphi_0 = 1\lambda, \tag{8.21}$$

we infer from the transformation rule (8.18) of φ that the little group of φ_0 is the diagonal $U(n)$ subgroup of $U(n) \times U(n)$. This shows that Yang–Mills theory on discrete space generates spontaneous symmetry breaking and thus we have translated this appealing result of Connes and Lott [1] into the Hamilton formalism.

8.2. Yang–Mills theory on space–time × two-point space

In this second example we utilize the result of the previous example to construct a Yang–Mills theory with spontaneously broken symmetry on a four dimensional Minkowskian space-time. We assume that the space–time manifold M has the topology $M_3 \times \mathbb{R}$ where M_3 is a compact manifold. For this example let us take for M_3 the one point compactification of $\mathbb{R}^3$, i.e. $M_3 = S^3$. The algebra $\mathcal{A}$ is of the form

$$\mathcal{A} = C^\infty(M) \otimes (\mathbb{C}^{n\times n} \oplus \mathbb{C}^{n\times n}) . \tag{8.22}$$

The space-like part of the algebra is

$$\begin{aligned}\mathcal{A}_s &= C^\infty(S^3) \otimes (\mathbb{C}^{n\times n} \oplus \mathbb{C}^{n\times n}) \\ &= C^\infty(S^3) \otimes \mathcal{A}_{\mathrm{mat}}.\end{aligned} \tag{8.23}$$

For S^3 there is a K-cycle $(\mathcal{H}_3, D_3)$ over $C^\infty(S^3)$, where $\mathcal{H}_3$ denotes the square integrable spin-sections over S^3 and D_3 denotes the Dirac-operator on S^3, which leads to the usual de Rham algebra. The K-cycle $(\mathcal{H}_{\mathrm{mat}}, D_{\mathrm{mat}})$ has been specified in the previous example (the subscript "mat" is introduced in order to distinguish objects referring to the discrete part of the algebra from the other objects). Usually one obtains a K-cycle over an algebra which is a tensor product of two algebras by taking the product K-cycle of the K-cycles over the factor algebras. However, there is one difficulty in our case. For the definition of the operator D of the product K-cycle one needs a grading on one of the factor K-cycles. Since S^3 is odd-dimensional there is no such grading on the Clifford-bundle over S^3. On the other hand for the Clifford-bundle over $\mathbb{R} \times S^3$ there is grading given by $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. Thus we can take the product K-cycle $(\mathcal{H}, D)$ over $\mathcal{A}$ with

$$D = D_4 \otimes 1_{\text{mat}} + \gamma^5 \otimes D_{\text{mat}}, \qquad \mathcal{H} = \mathcal{H}_4 \oplus \mathcal{H}_{\text{mat}}, \tag{8.24}$$

where $D_4 = \gamma^\mu \partial_\mu$ denotes the Dirac operator on $M = \mathbb{R} \times S^3$ and $\mathcal{H}_4$ is the space of square integrable spin-sections over M. Since a Dirac operator on a manifold with topology $\mathbb{R} \times M_3$ can always be decomposed in a time-like part D_t and a space-like part D_3 the space-like K-cycle $(\mathcal{H}_s, D_s)$ over $\mathcal{A}_s$ is

$$D_s = D_3 \otimes 1_{\text{mat}} + \gamma^5 \otimes D_{\text{mat}}, \qquad \mathcal{H}_s = \mathcal{H}_3 \otimes \mathcal{H}_{\text{mat}}. \tag{8.25}$$

Again we choose for the $\mathcal{A}$-module $\mathcal{E} = \mathcal{A}$.

The connection $\mathcal{A} = \boldsymbol{A}_t + \boldsymbol{A}_s$ for this model is

$$\boldsymbol{A}_t = \begin{pmatrix} \boldsymbol{A}_{t1} & 0 \\ 0 & \boldsymbol{A}_{t2} \end{pmatrix}, \qquad \boldsymbol{A}_s = \begin{pmatrix} \boldsymbol{A}_{s1} & \phi \\ \phi^\dagger & \boldsymbol{A}_{s2} \end{pmatrix}. \tag{8.26}$$

$\boldsymbol{A}_t$ is the same as in the previous example but on the block-diagonal of $\boldsymbol{A}_s$ there are now the space-like parts of the conventional gauge connections $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, i.e. $\boldsymbol{A}_{s1}$ and $\boldsymbol{A}_{s2}$ are anti-Hermitian matrices multiplied with space-like one forms.

The corresponding curvature is

$$\boldsymbol{F}_{st} = \begin{pmatrix} \boldsymbol{F}_{st1} & \begin{matrix} \dot\phi\, dt + \boldsymbol{A}_{t1}(\mu + \phi) \\ + (\mu + \phi)\boldsymbol{A}_{t2} \end{matrix} \\ \begin{matrix} \dot\phi^\dagger\, \mathrm{d}t + (\mu^\dagger + \phi^\dagger)\boldsymbol{A}_{t1} \\ + \boldsymbol{A}_{t2}(\mu^\dagger + \phi^\dagger) \end{matrix} & \boldsymbol{F}_{st2} \end{pmatrix},$$

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_1 + \phi\mu^\dagger + \mu\phi^\dagger + \phi\phi^\dagger & \begin{matrix} \partial_i\phi\, \mathrm{d}x^i + \boldsymbol{A}_{s1}(\mu + \phi) \\ + (\mu + \phi)\boldsymbol{A}_{s2} \end{matrix} \\ \begin{matrix} \partial_i\phi^\dagger\, \mathrm{d}x^i + (\mu^\dagger + \phi^\dagger)\boldsymbol{A}_{s1} \\ + \boldsymbol{A}_{s2}(\mu^\dagger + \phi^\dagger) \end{matrix} & \phi^\dagger\mu + \mu^\dagger\phi + \phi^\dagger\phi \end{pmatrix}, \tag{8.27}$$

where $\boldsymbol{B}_i$, $i = 1, 2$ denotes the space-like curvature of $\boldsymbol{A}_i$, ∇_{si} is the corresponding covariant space-like derivative and

$$\boldsymbol{F}_{sti} = -\partial_t \boldsymbol{A}_{si}\, \mathrm{d}t + \nabla_{si} \boldsymbol{A}_{ti}. \tag{8.28}$$

Because of Connes' trace theorem we know that in this case the Dixmier trace is equivalent to an integration over S^3 and hence the Lagrange function is

$$
\begin{aligned}
L &= -\frac{1}{4}\mathrm{tr}\int \mathrm{d}^3x(F^\dagger F) \\
&= \frac{1}{2}\int \mathrm{d}^3x\Big(-V(\phi) + \mathrm{tr}[\boldsymbol{F}_{st1}^2 + \boldsymbol{F}_{st2}^2 - \boldsymbol{B}_1^2 - \boldsymbol{B}_2^2 \\
&\quad + (\dot\phi\gamma^0 + \boldsymbol{A}_{t1}(\mu+\phi) + (\mu+\phi)\boldsymbol{A}_{t2})(\dot\phi^\dagger\gamma^0 + (\mu^\dagger+\phi^\dagger)\boldsymbol{A}_{t1} + \boldsymbol{A}_{t2}(\mu^\dagger+\phi^\dagger)) \\
&\quad - (\partial_i\phi\gamma^i + \boldsymbol{A}_{s1}(\mu+\phi) + (\mu+\phi)\boldsymbol{A}_{s2}) \\
&\quad \times(\partial_i\phi^\dagger\gamma^i + (\mu^\dagger+\phi^\dagger)\boldsymbol{A}_{s1} + \boldsymbol{A}_{s2}(\mu^\dagger+\phi^\dagger))]\Big)
\end{aligned}
\tag{8.29}
$$

with $V(\phi)$ given by Eq. (8.10).

The canonical momenta for this system are

$$
\boldsymbol{E} = \begin{pmatrix} \boldsymbol{E}_1 & -\pi^\dagger \\ \pi & \boldsymbol{E}_2 \end{pmatrix}
\tag{8.30}
$$

with π defined in Eq. (8.12) and

$$
\boldsymbol{E}_i = \boldsymbol{F}_{sti}, \quad i = 1,2.
\tag{8.31}
$$

Thus we can determine the Hamiltonian $H_0 - G(\boldsymbol{A}_t)$ to be

$$
\begin{aligned}
H_0 = \int \mathrm{d}^3x\Big(&V(\phi) + \mathrm{tr}[\boldsymbol{E}_1^2 + \boldsymbol{E}_2^2 + \pi^\dagger\pi + \boldsymbol{B}_1^2 + \boldsymbol{B}_2^2 \\
&+ (\partial_i\phi\gamma^i + \boldsymbol{A}_{s1}(\mu+\phi) + (\mu+\phi)\boldsymbol{A}_{s2}) \\
&\times(\partial_i\phi^\dagger\gamma^0 + (\mu^\dagger+\phi^\dagger)\boldsymbol{A}_{s1} + \boldsymbol{A}_{s2}(\mu^\dagger+\phi^\dagger))]\Big).
\end{aligned}
\tag{8.32}
$$

Again the Gauß-law can be summarized as

$$
G(\boldsymbol{A}_t) = \int \mathrm{d}^3x\, \mathrm{tr}[\boldsymbol{E}(D_s + \boldsymbol{A}_s)\boldsymbol{A}_t + \boldsymbol{E}\boldsymbol{A}_t(D_s + \boldsymbol{A}_s)]\,.
\tag{8.33}
$$

The phase-space variables transform as follows:

$$
\delta\boldsymbol{E}_i = \Lambda_i\boldsymbol{E}_i - \boldsymbol{E}_i\Lambda_i, \quad i = 1,2.
\tag{8.34}
$$

The transformation rule for the fields π and ϕ are determined by Eq. (8.16). By shifting ϕ to $\varphi = \phi + \mu$ we obtain a field which transforms homogeneously under gauge transformations. For $\varphi^\dagger\varphi = 1\lambda^2$ the potential is minimized and thus the symmetry is spontaneously broken. In the gauge

$$
\varphi = 1\lambda,
\tag{8.35}
$$

we see that $\boldsymbol{A}_+ = \boldsymbol{A}_1 + \boldsymbol{A}_2$ correspond to the massless modes of the gauge fields and $\boldsymbol{A}_- = \boldsymbol{A}_1 - \boldsymbol{A}_2$ correspond to the massive modes.

9. Conclusions

We have derived the Hamilton formalism for Yang–Mills theory in non-commutative geometry. For this purpose we exploited the special structure of $\mathcal{A} = C(I, \mathcal{A}_s)$ which seems to be very natural since the topology of space–time in the conventional Hamilton formalism is $M = \mathbb{R} \times \Sigma$. The first step was to show that the structure of the algebra together with an appropriately chosen K-cycle allows to identify the time-like part of the generalized differential algebra. Thus the notion of time obtains a well defined meaning in this context.

The next step was to introduce the non-commutative generalization of integration over space-like surfaces via the Dixmier trace. This opened the possibility to apply the formalism to Minkowskian space–time by abandoning the ellipticity of the operator D of the K-cycle $(\mathcal{H}, D)$ over $\mathcal{A}$ but maintaining the ellipticity of the space-like part D_s of D. However, in this case one is restricted to the non-commutative counterpart of integration over space-like surfaces. For the definition of Lagrange functions and Hamilton functions integration over space-like surfaces is sufficient. For the definition of actions one may use a hybrid formalism, i.e. one performs integration over the (possibly non-commutative) space-like surface via Dixmier trace and for the time variable one uses conventional integration. The structure $C(I, \mathcal{A}_s)$ of the algebra ensures that this is possible.

For the definition of the Poisson bracket we had to make some additional assumptions which we introduced at the end of Section 5. Especially the assumption which allowed us to define the adjoint of the operator d seems to be a brute force assumption. Although all assumptions we made are fulfilled for the examples we presented, a finer criterion for the existence of an adjoint of d seems to be desirable.

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